

On stochasticity in nearly-elastic systems

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Abstract

Nearly-elastic model systems with one or two degrees of freedom are considered: the system is undergoing a small loss of energy in each collision with the "wall". We show that instabilities in this purely deterministic system lead to stochasticity of its long-time behavior. Various ways to give a rigorous meaning to the last statement are considered. All of them, if applicable, lead to the same stochasticity which is described explicitly. So that the stochasticity of the long-time behavior is an intrinsic property of the deterministic systems.

Keywords: Averaging principle; Hamiltonian flows; Markov processes on graphs; chaotic systems.

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1 Introduction

Consider the one-dimensional motion of a unit-mass particle in a smooth potential $F(q)$ in an interval $[a_1, a_2]$ with elastic reflection at a_1 and a_2 . Let $F'(a_1) > 0$ and $F'(a_2) < 0$ (Fig.1(a), the potential $F(q)$ is shown there as the bold-face line): if $q \in [a_1, a_2)$ and $p > 0$ or $q \in (a_1, a_2]$ and $p < 0$, the trajectory starting at (q, p) moves according to equation $\ddot{q}(t) = -F'(q(t))$; the trajectory jumps instantaneously from (a_i, p) to $(a_i, -p)$, if $i = 2$ and $p > 0$ or if $i = 1$ and $p < 0$. If the initial velocity is large enough, the particle hits both "walls" a_1 and a_2 and performs periodic oscillations shown in Fig.1(b). Let now the walls be not absolutely elastic. It is natural to assume in certain situations that the loss of energy is proportional to the speed at the collision point with the wall: if the particle hits a_i with a speed v , it is instantly reflected with the speed $-v(1 - \varepsilon c_i(v))$, $i \in \{1, 2\}$. Here $c_i(v)$ are positive smooth functions, $0 < \varepsilon \ll 1$. The coefficient $(1 - \varepsilon c_i(v))$ is called the coefficient of restitution. Denote by $q^\varepsilon(t)$ the position at time t of the particle performing nearly-elastic motion, $p^\varepsilon(t) = \dot{q}^\varepsilon(t)$. It is

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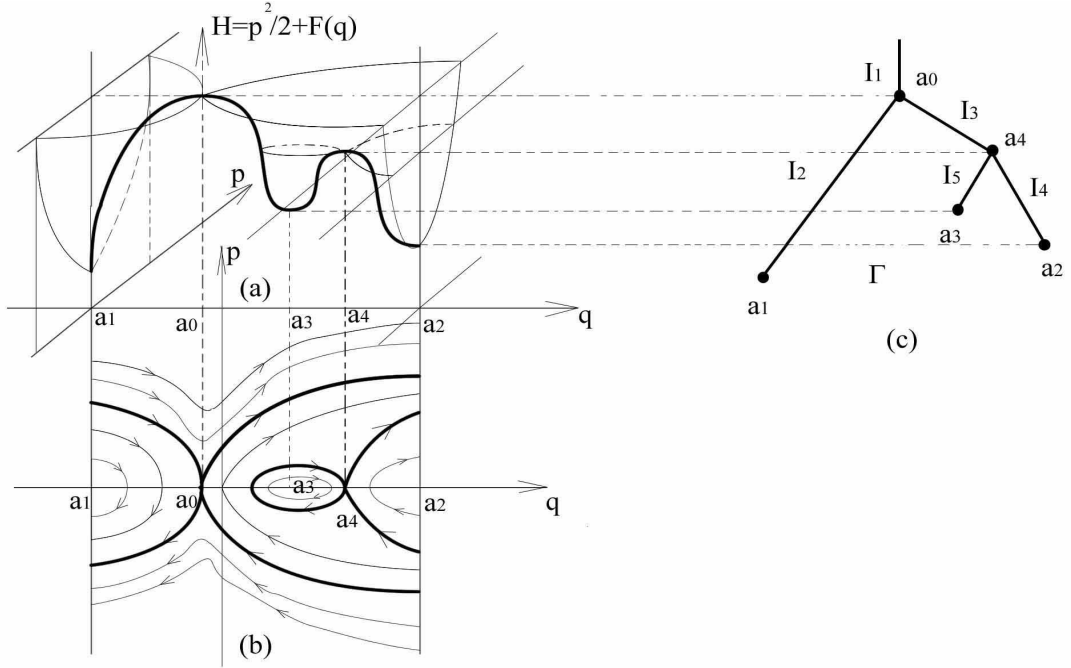


Fig. 1: The 1-dimensional mechanical model

clear that, for each $\varepsilon > 0$ and t large enough, $q^\varepsilon(t)$ will be situated in an arbitrary small neighborhood of one of the points a_1, a_2, a_3 (We assume that the potential $F(q)$ has the form shown in Fig.1(a) and a_3 is the unique local minimum inside $[a_1, a_2]$).

To be specific, assume that the initial speed $\dot{q}^\varepsilon(0) = p^\varepsilon(0) = v_0$ is large enough : $v_0 > \sqrt{2 \max F(q)}$. Then $q^\varepsilon(t)$, for ε small enough, hits both points a_1 and a_2 , but $\lim_{t \rightarrow \infty} q^\varepsilon(t) = q_\infty^\varepsilon$ exists and $q_\infty^\varepsilon = a_1$ or $q_\infty^\varepsilon = a_2$. Which of these two points appears as $\lim_{t \rightarrow \infty} q^\varepsilon(t)$ depends on ε in a very sensitive way so that $\lim_{\varepsilon \downarrow 0} q_\infty^\varepsilon = \lim_{\varepsilon \downarrow 0} \lim_{t \rightarrow \infty} q^\varepsilon(t)$ does not exist. We show that the final position q_∞^ε converges, in a certain sense, as $\varepsilon \downarrow 0$ to a random variable distributed between the points a_1 and a_2 . There are various ways to make the last statement rigorous. But they all lead to the same distribution between a_1 and a_2 , so that the stochasticity of the system as $\varepsilon \ll 1$ is an intrinsic property of the system.

The perturbed system $q^\varepsilon(t)$ for $0 < \varepsilon \ll 1$ has fast and slow components. The fast component consists of the motion along the non-perturbed trajectory. To describe the slow component, consider the graph Γ obtained after identification of points of each connected component of every level set of the Hamiltonian $H(q, p) = \frac{p^2}{2} + F(q)$ (Fig.1(c)). Denote by $Y : \square \rightarrow \Gamma$ the identification map of the phase space $\square = \{(q, p) \in \mathbb{R}^2 : a_1 \leq q \leq a_2\}$ of our system on Γ . The slow component of the motion is $Y(q^\varepsilon(t), \dot{q}^\varepsilon(t))$ (compare with [6], Ch.8). Number the edges of the graph ($\Gamma = I_1 \cup I_2 \cup \dots \cup I_5$ in the Fig.1(c)). Then $Y(q, p) = (H(q, p), K(q, p))$, $(q, p) \in \square$, where $K(q, p)$ is

the number of the edge containing $Y(q, p)$ and $H(q, p) = \frac{p^2}{2} + F(q)$. The pair (H, K) form a global coordinate system on Γ , and the slow component is the motion $Y_t^\varepsilon = (H(q^\varepsilon(t), \dot{q}^\varepsilon(t)), K(q^\varepsilon(t), \dot{q}^\varepsilon(t)))$ on Γ .

We prove that, in a certain sense, the rescaled slow motion $Y_{t/\varepsilon}^\varepsilon$ converges weakly to a stochastic process Y_t on Γ . Inside the edges, Y_t is a deterministic motion and the convergence follows from more or less standard averaging principle (see, for instance, [1], Ch.10). The stochasticity appears due to a branching at the interior vertices of Γ (compare [2]).

Note that in the case of system shown in Fig.1, the phase trajectory $(q^\varepsilon(t), \dot{q}^\varepsilon(t))$ with $\dot{q}_0^\varepsilon = v_0 > \sqrt{2 \max_{a_1 \leq q \leq a_2} F(q)}$ never enters the domain $\{(q, p) \in \Pi : Y(q, p) \in I_5\}$, so that if the damping of the system occurs just on the walls a_1 and a_2 , the stochasticity of the limiting slow motion is concentrated at the vertex corresponding to the absolute maximum of the potential. Therefore we will consider in more detail the case when $F(q)$ has just one local maximum on $[a_1, a_2]$. If the potential has many wells and the system is losing energy not just at the walls a_1, a_2 , one should take into account the whole graph Γ even if the initial speed is large.

In the next two sections we consider a model problem where the limiting slow motion inside the edges is just a motion with constant speed. This system has some damping not just in the ends of the interval $[a_1, a_2]$ and models systems with multi-well Hamiltonian. To give a rigorous meaning to the statement that the slow motion converges to a stochastic process, we, first, perturb the system $q^\varepsilon(t)$ by a stochastic perturbation of a small intensity $\delta \ll 1$. Then in this already stochastic system $q^{\varepsilon, \delta}(t)$, we consider the slow component $Y_t^{\varepsilon, \delta} = Y(q^{\varepsilon, \delta}(t), \dot{q}^{\varepsilon, \delta}(t))$. We show that $Y_{t/\varepsilon}^{\varepsilon, \delta}$ converges, as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, in a certain sense to a stochastic process Y_t on Γ . This double limit $\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} Y_t^{\varepsilon, \delta} = Y_t$ exists and is the same for a broad class of stochastic regularizations. It is in this sense that the slow component of the purely deterministic system $q^\varepsilon(t)$ should be approximated by the stochastic process Y_t on Γ .

In section 4 we come back to the problem mentioned in the beginning of this section.

Of course, one can consider similar questions for systems with more than one degree of freedom. Let, say, $F(q)$, $q \in \mathbb{R}^2$ be as shown in Fig.2. The particle of a unit mass moves inside a convex domain G with a smooth boundary ∂G according to the equation $\ddot{q}_t = -\nabla F(q_t)$ and undergoes an instantaneous mirror reflection on the boundary. If the collisions with the wall ∂G are absolutely elastic, the particle will move forever on the energy surface $H(p, q) = \frac{|p|^2}{2} + F(q) = H_0$. But if the particle loses energy in each collision similar to the case of one degree of freedom, it can, eventually, be found near one of the local minima of $F(q)$ on ∂G (points O_1 and O_2 in Fig.2). We consider many-degrees-of-freedom problems for model potentials in Section 5.

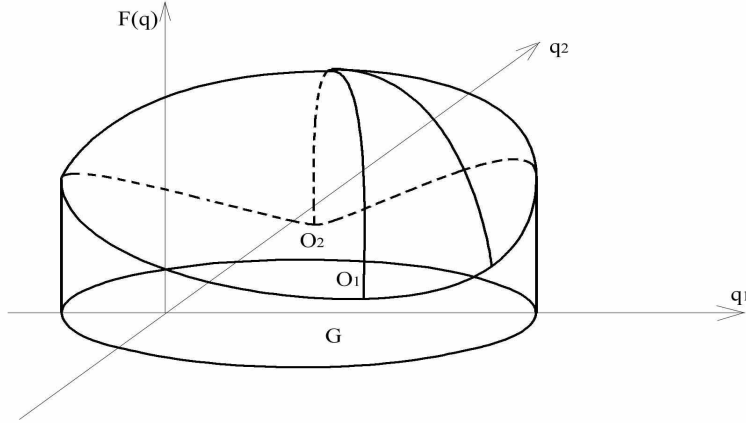


Fig. 2: A 2-dimensional problem

2 One-degree-of-freedom Model Problems

Consider a model of the system with several potential wells. Suppose a particle of unit mass moves freely in an interval $[q_1, q_n]$ with elastic reflection at the ends of the interval if the initial velocity is large enough. Let a finite number of points $q_2, q_3, \dots, q_{n-1} \in (q_1, q_n)$ and $0 < p_2 < \dots < p_{n-1}$ are given such that if the particle starts with a velocity p_0 at a point $q_0 \in [q_1, q_n]$ then it moves freely in $[q_{i_-(p_0, q_0)}, q_{i_+(p_0, q_0)}]$ with instantaneous reflection in the ends of this interval, where $i_- = i_-(p_0, q_0)$ and $i_+ = i_+(p_0, q_0)$ are defined by the conditions (see Fig.3):

$$q_{i_-} < q_0 < q_{i_+} \quad , \quad p_0 \leq \min(p_{i_-}, p_{i_+}) \quad , \quad p_0 > p_k \quad \text{for } k \in \{i : q_i \in (q_{i_-}, q_{i_+})\}.$$

The energy $\frac{p_0^2}{2}$ is preserved in this system. This is an approximation of the motion in a potential which has sharp maxima at points q_i and is close to a constant between q_i and q_{i+1} . Just for brevity, we assume that the walls between the maxima have the same depth.

Assume again now that the collisions with the walls are not absolutely elastic: if the particle hits the wall at the point q_k with a speed p , $|p| \leq p_k$, it is reflected from q_k with the speed $-p(1 - \varepsilon c_k(p))$, where $c_k(\cdot)$ are defined as before but now $(1 - \varepsilon c_k(p))$ represents the coefficient of restitution for the wall $\{(q, p) \in \mathbb{R}^2 : 0 < p \leq p_k, q = q_k\}$, and $0 < \varepsilon \ll 1$. Then for each $\varepsilon > 0$, the velocity tends to zero as $t \rightarrow \infty$. On each velocity level p_k , $p_k < p_0$, phase trajectory goes to one of the two "wells" separated by q_k . Corresponding phase space $\square = \{(q, p) \in \mathbb{R}^2, q_1 \leq q \leq q_n\}$ is shown in Fig.3 for $n = 4$. Notice that actually each interior wall in the phase space \square consists of two sides (the broken line represents another side in Fig.3). For simplicity we will identify the two sides in our investigation and we assume that the coefficients $c_k(\cdot)$ is the same for both

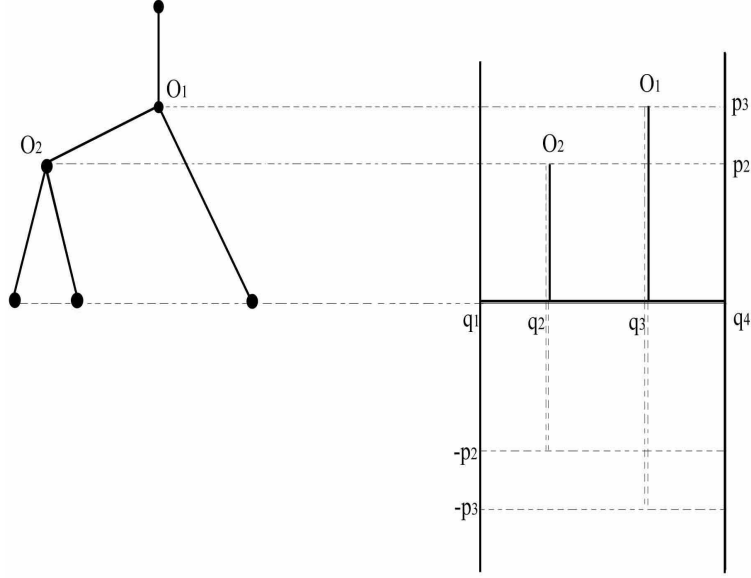


Fig. 3: 1-dimensional model problem

faces of the wall $q = q_k$.

We will see that the "choice of the well" at each q_k is very sensitive to ε as $\varepsilon \downarrow 0$, and the behavior of the trajectory becomes, actually, stochastic in a certain sense. Now we will give the exact meaning to this statement and describe the limiting stochastic process.

Let us, first, consider the case when the model system has only two trapping wells. An example of such a model is shown in Fig.4. A particle with unit mass starts its motion from a point $x = (q_0, p_0)$ in the phase space \square and has instantaneous reflection each time when it hits the boundary $q = -a_1$ (corresponding to well \mathcal{E}_1) or $q = a_2$ (corresponding to well \mathcal{E}_2). Each time when the particle hits the boundary $q = -a_1$ or $q = a_2$ it will be reflected and move to the point $(-1, -p(1 - \varepsilon c_1(p)))$ or $(1, -p(1 - \varepsilon c_2(p)))$, respectively. The coefficient of restitution for the middle wall (notice that it has two sides) $q = 0$ is $1 - \varepsilon c_3(\cdot)$. After the particle enters one of the wells \mathcal{E}_1 or \mathcal{E}_2 , it hits corresponding boundaries and loses energy according to the coefficient of restitution of the corresponding boundary. We denote such a motion by $X_t^{\varepsilon, x} = X_t^\varepsilon = (q^\varepsilon(t), p^\varepsilon(t))$, where $(q^\varepsilon(0), p^\varepsilon(0)) = (q_0, p_0) = x$. The superscript x represents the starting point of the motion. Here and henceforth, when we use the notation X_t^ε , it means that we have omitted the superscript x . Put $H^\varepsilon(t) = \frac{(p^\varepsilon(t))^2}{2}$. We rescale time and let $\tilde{X}_t^\varepsilon = X_{t/\varepsilon}^\varepsilon$. As before, we consider the slow component of \tilde{X}_t^ε which is the projection $Y_t^\varepsilon = (H(\tilde{X}_t^\varepsilon), K(\tilde{X}_t^\varepsilon))$ of \tilde{X}_t^ε onto the graph Γ . Here $H(\tilde{X}_t^\varepsilon) = H^\varepsilon(t/\varepsilon)$. Since the coefficient of restitution depends only on the magnitude of velocity but not its direction,

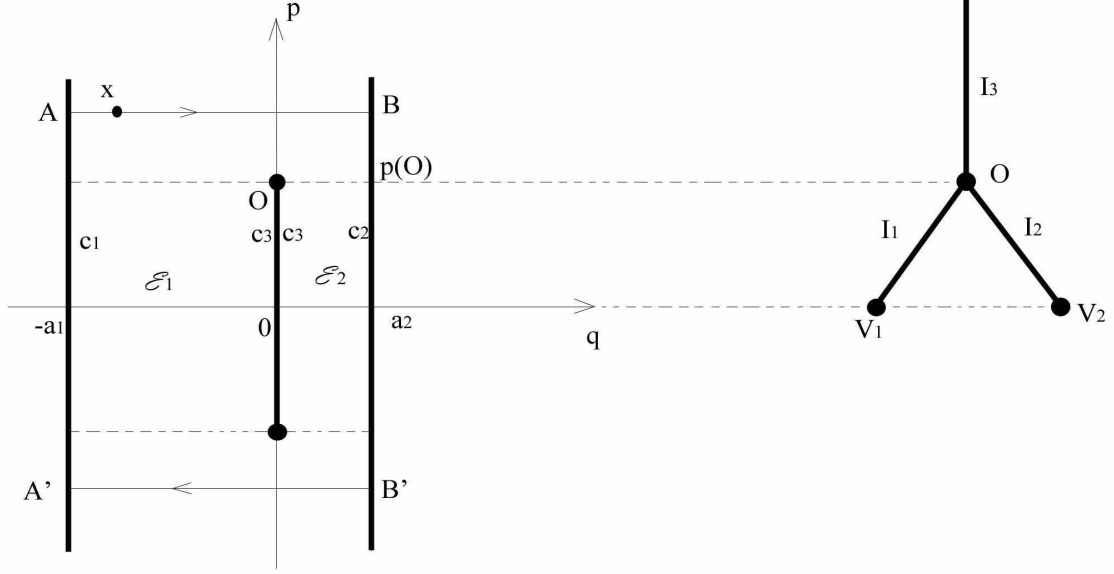


Fig. 4: The case when the model system has only two wells

i.e. $c_i(-p) = c_i(p)$, $i = 1, 2$, we can write $c_i(-p) = c_i(p) = c_i(\sqrt{2H})$. Now we prove:

Lemma 2.1 *Within each edge of the graph Γ , as $\varepsilon \downarrow 0$, the first component of the process Y_t^ε , i.e. $H(\tilde{X}_t^\varepsilon) = \frac{(p^\varepsilon(t/\varepsilon))^2}{2}$, converges uniformly to a deterministic motion $H(t)$ which satisfies the differential equation*

$$\frac{dH}{dt} = -2 \frac{c_1(\sqrt{2H}) + c_2(\sqrt{2H})}{T_3(H)} H, \quad H(0) = H_0 \quad \text{on } I_3, \quad (2.1)$$

and

$$\frac{dH}{dt} = -2 \frac{c_1(\sqrt{2H}) + c_3(\sqrt{2H})}{T_1(H)} H, \quad \text{on } I_1, \quad (2.2)$$

$$\frac{dH}{dt} = -2 \frac{c_2(\sqrt{2H}) + c_3(\sqrt{2H})}{T_2(H)} H, \quad \text{on } I_2, \quad (2.3)$$

respectively. Here $T_3(H) = \frac{2(a_1 + a_2)}{\sqrt{2H}}$, $T_1(H) = \frac{2a_1}{\sqrt{2H}}$, $T_2(H) = \frac{2a_2}{\sqrt{2H}}$ are the corresponding periods of motion on phase picture \square along the energy level H for each edge of Γ .

Proof: Let us prove, for example, (2.1). The proofs of (2.2) and (2.3) are exactly similar to the proof of (2.1). We prove this result by using a slight modification of the standard method of justification of the averaging principle (compare with [1], Ch.10). First of all, we view the whole contour (see Fig.4) $A \rightarrow B \rightarrow B' \rightarrow A' \rightarrow A$ on which the particle has a fixed amount of energy H as a loop G_H (i.e. identify A with A' and

B with B'). One can introduce coordinates $(Q_{t/\varepsilon}^\varepsilon, H_{t/\varepsilon}^\varepsilon)$ to describe the motion of the particle on this circle. Here $Q_{t/\varepsilon}^\varepsilon = q_{t/\varepsilon}^\varepsilon + 2(k-1)(a_1 + a_2)$ if the particle is the k -th time on the upper half plane of the phase space, and $Q_{t/\varepsilon}^\varepsilon = -q_{t/\varepsilon}^\varepsilon + 2(k-1)a_1 + 2ka_2$ if the particle is the k -th time on the lower half plane of the phase space. We identify $Q_{t/\varepsilon}^\varepsilon$ with $Q_{t/\varepsilon}^\varepsilon + 2k(a_1 + a_2)$, $k \in \mathbb{Z}$. The motion $(Q_{t/\varepsilon}^\varepsilon, H_{t/\varepsilon}^\varepsilon)$ has a fast component $Q_{t/\varepsilon}^\varepsilon$ and a slow component $H_{t/\varepsilon}^\varepsilon$ and they satisfy the following equation

$$\begin{cases} \dot{Q}_{t/\varepsilon}^\varepsilon = \frac{1}{\varepsilon} \sqrt{2H_{t/\varepsilon}^\varepsilon}, \\ \dot{H}_{t/\varepsilon}^\varepsilon = -[\Delta(Q_{t/\varepsilon}^\varepsilon - a_2) + \Delta(Q_{t/\varepsilon}^\varepsilon + a_1)](2c(Q_{t/\varepsilon}^\varepsilon, \sqrt{2H_{t/\varepsilon}^\varepsilon}) - \varepsilon c^2(Q_{t/\varepsilon}^\varepsilon, \sqrt{2H_{t/\varepsilon}^\varepsilon}))H_{t/\varepsilon}^\varepsilon. \end{cases} \quad (2.4)$$

Here $\Delta(\cdot)$ is the Dirac δ -function and $c(Q_{t/\varepsilon}^\varepsilon, \sqrt{2H_{t/\varepsilon}^\varepsilon}) = c_1(\sqrt{2H_{t/\varepsilon}^\varepsilon})$ if $Q_{t/\varepsilon}^\varepsilon = -a_1$ and $c(Q_{t/\varepsilon}^\varepsilon, \sqrt{2H_{t/\varepsilon}^\varepsilon}) = c_2(\sqrt{2H_{t/\varepsilon}^\varepsilon})$ if $Q_{t/\varepsilon}^\varepsilon = a_2$.

Now we introduce an auxiliary function $u = u(Q, H)$ satisfying the following differential equation

$$\sqrt{2H} \frac{\partial u}{\partial Q} = A - \bar{A}, \quad Q \in G_H \quad (2.5)$$

where

$$A = -[\Delta(Q - a_2) + \Delta(Q + a_1)](2c(Q, \sqrt{2H}) - \varepsilon c^2(Q, \sqrt{2H}))H,$$

and

$$\bar{A} = -2 \frac{c_1(\sqrt{2H}) + c_2(\sqrt{2H}) - \varepsilon(c_1^2(\sqrt{2H}) + c_2^2(\sqrt{2H}))}{T_3(H)}$$

is the average of A along the loop $A \rightarrow B \rightarrow B' \rightarrow A' \rightarrow A$.

One can check the following properties of $u(Q, H)$: (i) the differential equation (2.5) describing u really has a well-defined solution $u(Q, H)$ on the loop G_H ; (ii) the solution $u(Q, H)$ is uniformly bounded together with its first derivatives for $Q \in G_H$, $H \in [H(O), M]$ for any $M \in (H(O), \infty)$.

Now, by the fundamental theorem of calculus, we have:

$$\begin{aligned} & u(Q_{t/\varepsilon}^\varepsilon, H_{t/\varepsilon}^\varepsilon) - u(Q_0^\varepsilon, H_0^\varepsilon) \\ &= \int_0^t \frac{\partial u}{\partial Q} \dot{Q}_{s/\varepsilon}^\varepsilon ds + \int_0^t \frac{\partial u}{\partial H} \dot{H}_{s/\varepsilon}^\varepsilon ds \\ &= \frac{1}{\varepsilon} \int_0^t (A - \bar{A}) ds + O(1). \end{aligned}$$

Therefore we see that $\max_{0 \leq t \leq T} \left| \int_0^t (A - \bar{A}) ds \right| \leq C\varepsilon$ for some positive $C > 0$.

Now, we compare the two differential equations:

$$\dot{H}_{t/\varepsilon}^\varepsilon = -[\Delta(Q_{t/\varepsilon}^\varepsilon - a_2) + \Delta(Q_{t/\varepsilon}^\varepsilon + a_1)](2c(Q_{t/\varepsilon}^\varepsilon, \sqrt{2H_{t/\varepsilon}^\varepsilon}) - \varepsilon c^2(Q_{t/\varepsilon}^\varepsilon, \sqrt{2H_{t/\varepsilon}^\varepsilon}))H_{t/\varepsilon}^\varepsilon, \quad H_0^\varepsilon = H_0;$$

$$\frac{dH}{dt} = -2 \frac{c_1(\sqrt{2H}) + c_2(\sqrt{2H})}{T_3(H)} H, \quad H(0) = H_0.$$

Let $m(T) = \max_{0 \leq t \leq T} |H_{t/\varepsilon}^\varepsilon - H(t)|$. We have, that

$$\begin{aligned} & |H_{t/\varepsilon}^\varepsilon - H(t)| \\ &= \left| \int_0^t (AH_{s/\varepsilon}^\varepsilon - \bar{A}H(s)) ds \right| + 2\varepsilon B \\ &\leq \int_0^t |A - \bar{A}| H_{s/\varepsilon}^\varepsilon ds + \int_0^t \bar{A} |H_{s/\varepsilon}^\varepsilon - H(s)| ds + 2\varepsilon B. \end{aligned}$$

The last inequality implies that $m(T) \leq (CD + 2B)\varepsilon + E \int_0^T m(s) ds$, where B is the uniform bound for $\frac{c_1^2(\sqrt{2H}) + c_2^2(\sqrt{2H})}{T_3(H)}$, D is the uniform bound for $H_{s/\varepsilon}^\varepsilon$, and E is the uniform bound for \bar{A} . By using Gronwall's inequality, we see that $m(T) \leq (CD + 2B)\varepsilon e^{ET}$, i.e. $\lim_{\varepsilon \downarrow 0} \max_{0 \leq t \leq T} |H_{t/\varepsilon}^\varepsilon - H(t)| = 0$. ■

Notice that in this case $T_i(H) \sim H^{-\frac{1}{2}}$ as $H \rightarrow H(O)$, this means that Y_t^ε will enter the interior vertex O of the graph Γ in finite time. Thus one might ask what is the behavior of the motion Y_t^ε at the interior vertex O . First, we regularize this problem by considering small stochastic perturbation of the initial conditions in our model.

Lemma 2.2 *Let $\mathcal{U}(x, \delta) = \{y \in \square : |x - y| < \delta\}$, $x \in \square$, $\delta > 0$, $\mathcal{U}_i^\varepsilon = \{x \in \square : X^{\varepsilon, x} \text{ eventually enters the well } \mathcal{E}_i\}$, $i \in \{1, 2\}$ (see Fig.5), where $X^{\varepsilon, x}$ is the perturbed trajectory starting at $X_0^{\varepsilon, x} = x \in \square$. Assume that $H_0 = H(x) > H(O)$. Then*

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{\mu(\mathcal{U}_1^\varepsilon \cap \mathcal{U}(x, \delta))}{\mu(\mathcal{U}_2^\varepsilon \cap \mathcal{U}(x, \delta))} = \frac{c_1(H(O))}{c_2(H(O))},$$

where μ is the Lebesgue measure in \mathbb{R}^2 .

Proof: Without loss of generality one can assume that $p_0 > 0$. We see, from Fig.5, that $\mathcal{U}(x, \delta)$ is covered by narrow shaded or white strips, where shaded strips belong to $\mathcal{U}_1^\varepsilon$, and white strips belong to $\mathcal{U}_2^\varepsilon$. For a fixed x , consider the nearest to x shaded and white strip (ordered as a pair, shaded on top). The upper p -level of the shaded strip of the pair is denoted by a . And the lower p -level of the shaded strip (also the upper p -level of the white strip in the pair) by b . Let the lower p -level of the white strip in the

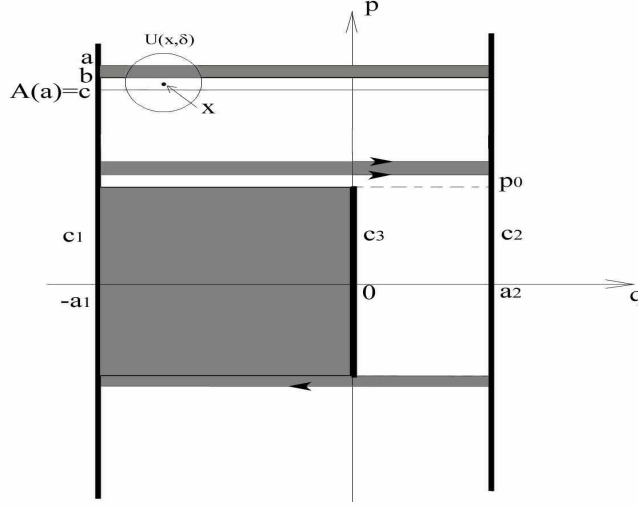


Fig. 5: Regularization by stochastic perturbation of the initial condition

pair be c . Let $n = n(\varepsilon, x)$ be the number of collisions that the particle made with either of the walls $q = -a_1$ or $q = a_2$ before entering one of the wells.

Let us first estimate the p -width of the shaded and white strips. Put

$$A(a) = a(1 - \varepsilon c_1(a))(1 - \varepsilon c_2(a(1 - \varepsilon c_1(a)))) = a(1 - \varepsilon f_1^\varepsilon(a) + \varepsilon^2 f_2^\varepsilon(a))$$

where $f_1^\varepsilon(a) = c_1(a) + c_2(1 - \varepsilon c_1(a))$ and $f_2^\varepsilon(a) = c_1(a)c_2(1 - \varepsilon c_1(a))$. Then the boundaries of the next shaded and white strip has p -height $A(a) = c, A(b), A(c)$, respectively.

We see that

$$|c - a| = |A(a) - a| = \varepsilon a |f_1^\varepsilon(a) - f_2^\varepsilon(a)|,$$

So that

$$M_1 \varepsilon \leq |c - a| \leq M_2 \varepsilon$$

for some $M_1, M_2 > 0$. Therefore, $n = n(x, \varepsilon) \sim O(\frac{1}{\varepsilon})$.

We have

$$\begin{aligned} A(a) - A(b) &= a - b - \varepsilon(a f_1^\varepsilon(a) - b f_1^\varepsilon(b)) + \varepsilon^2(a f_2^\varepsilon(a) - b f_2^\varepsilon(b)) \\ &= a - b - \varepsilon(u_1^\varepsilon(b)(a - b) + u_2^\varepsilon(\xi)(a - b)^2) + \varepsilon^2(v_1^\varepsilon(b)(a - b) + v_2^\varepsilon(\eta)(a - b)^2) \\ &= (a - b)(1 - \varepsilon u_1^\varepsilon(b) + O(\varepsilon^2)) \end{aligned}$$

where $u_i, v_i, i \in \{1, 2\}$ are bounded smooth functions, and $\xi, \eta \in [b, a]$.

Similarly,

$$A(b) - A(c) = (b - c)(1 - \varepsilon u_1^\varepsilon(c) + O(\varepsilon^2)).$$

These inequalities imply that

$$\begin{aligned}
& \frac{A(a) - A(b)}{A(b) - A(c)} \\
&= \frac{a - b}{b - c} \left(\frac{1 - \varepsilon u_1^\varepsilon(b) + O(\varepsilon^2)}{1 - \varepsilon u_1^\varepsilon(c) + O(\varepsilon^2)} \right) \\
&= \frac{a - b}{b - c} \left(1 - \frac{\varepsilon(u_1^\varepsilon(b) - u_1^\varepsilon(c)) + O(\varepsilon^2)}{1 - \varepsilon u_1^\varepsilon(c) + O(\varepsilon^2)} \right) \\
&= \frac{a - b}{b - c} (1 + O(\varepsilon^2)).
\end{aligned}$$

Thus after $n = n(\varepsilon, x) \sim O(\frac{1}{\varepsilon})$ steps the change of ratio of the p -width of the nearby shaded and white strips is asymptotically $(1 + O(\varepsilon^2))^{n(\varepsilon, x)} \sim 1$, and our lemma then immediately follows from the definition of the sets $\mathcal{U}(x, \delta)$, $\mathcal{U}_1^\varepsilon$ and $\mathcal{U}_2^\varepsilon$. ■

In order to include our result in the framework of weak convergence of stochastic processes in the space of continuous functions, we still need the following technical construction. Notice that, for fixed $\varepsilon > 0$, the function $H^\varepsilon(t/\varepsilon)$ as a function of t , is a step function. We can assume that it is continuous from the right and having limit from the left. We now connect neighboring discontinuity points of the graph of $H^\varepsilon(t/\varepsilon)$ by straight line segments. The resulting trajectory is denoted by $\hat{H}^\varepsilon(t)$. For fixed $T > 0$, we see that $\hat{H}^\varepsilon(t) \in C_{0T}$. For a positive constant $C > 0$ and all $0 < t < T$, we have

$$|H^\varepsilon(t/\varepsilon) - \hat{H}^\varepsilon(t)| < C\varepsilon. \quad (2.6)$$

Since the time distance between two discontinuity points is bounded from below by the inverse of the initial velocity, the slope of all the line segments of the graph of $\hat{H}^\varepsilon(t)$ is bounded uniformly in $\varepsilon \in (0, 1]$. Thus the family of functions $\{\hat{H}^\varepsilon(t)\}_{\varepsilon > 0}$ is an equicontinuous family in C_{0T} . Also, it is uniformly bounded for $0 < t < T$ and $\varepsilon \in (0, 1]$. Thus by Ascoli-Arzelà Theorem the family $\{\hat{H}^\varepsilon(t)\}_{\varepsilon > 0}$ is compact in C_{0T} with uniform topology.

Let ξ_δ be a two-dimensional random variable with a continuous density $f_\delta(x)$ such that $f_\delta(x) > 0$ for $|x| < \delta$ and $f_\delta(x) = 0$ for $|x| \geq \delta$. Let $X_t^{\varepsilon, \delta} = (p_t^{\varepsilon, \delta}, q_t^{\varepsilon, \delta})$ be the trajectory of the nearly-elastic motion starting at $X_0^{\varepsilon, \delta} = x + \xi_\delta$ (the point $x = (q_0, p_0) \in \square$ is fixed, and we excluded it from the notation). Put $\tilde{X}_t^{\varepsilon, \delta} = X_{t/\varepsilon}^{\varepsilon, \delta}$.

Consider a stochastic process $Y_t = (H(t), K(t))$, $0 < t < T$, on the graph Γ defined by the conditions: $Y_0 = Y(q_0, p_0)$; $H(t)$ is a deterministic motion inside each edge of Γ satisfying equations (2.1)-(2.3) respectively; if $Y_0 = Y(q_0, p_0) \in I_3$, the trajectory Y_t reaches the interior vertex $O \in \Gamma$ in a finite time, instantaneously leaves O and enters the

edges I_1 or I_2 with probabilities $p_1 = \frac{c_1(H(O))}{c_1(H(O)) + c_2(H(O))}$, $p_2 = \frac{c_2(H(O))}{c_1(H(O)) + c_2(H(O))}$, respectively. The conditions listed above define the process Y_t in a unique way (in the sense of distributions).

Taking into account (2.6), the remark concerning compactness and Lemmas 2.1 and 2.2, we get the following

Theorem 2.1 *The process $\hat{Y}_t^{\varepsilon, \delta} = (\hat{H}(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$ on Γ converges weakly in the space of continuous functions $[0, T] \rightarrow \Gamma$ equipped with the uniform topology to the process $Y_t = (H(t), K(t))$ when, first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$.*

We now discuss the case when the model problem has more than two energy trapping wells. It turns out that in the multi-well case, the perturbation of the initial condition, in general, will *not* lead to a regularization of the problem: the limit of $Y_t^{\varepsilon, \delta}$ as $\varepsilon \downarrow 0$ may not exist.

Consider an example as shown in Fig.6. There are 3 trapping wells, 1, 2 and 3. Wells 2 and 3 are separated on a p -level p_b^* and they combined together are separated from well 1 on a p -level p_a^* . $p_a^* > p_b^*$. We call the wells 2 and 3 combined together well 4. Well 1 and well 4 combined together is called well 5. Suppose the final strip A as drawn in the picture is for those initial values that finally enter well 4. The boundary restitution coefficients are positive constants c_1 and $c_2 = c_3 = c_4 = c_5 = c (\neq c_1)$ as shown in Fig.6.

For each $\varepsilon > 0$, the shadowed strip A on the top of wells 1 and 4 is moved by the nearly-elastic dynamics to the well 4. The strip A has the width $c\varepsilon$. The well 4 between p -levels p_b^* and p_a^* is also covered by strips of width $c\varepsilon$ (suppose that ε is such that an integer number of such strips is situated between p_b^* and p_a^*). The highest strip B in the wall 4 is moved by our dynamics (as a whole) alternatively either to well 2 or to well 3 as $\varepsilon \downarrow 0$. But the strip A in one step (one reflection from the right exterior wall) is going to the strip B . Therefore, for the sequence of $\varepsilon \downarrow 0$ for which $\frac{p_a^* - p_b^*}{c\varepsilon}$ is an integer, the nearly-elastic trajectory starting from any $x = (q_0, p_0)$, $p_0 > p_a^*$, for t large enough is distributed alternatively either between walls 1 and 2 or between walls 1 and 3. This means that the limiting distribution for $X_t^{\varepsilon, \delta}$ as $\varepsilon \downarrow 0$ does not exist, and the problem cannot be regularized by random perturbations of the initial point.

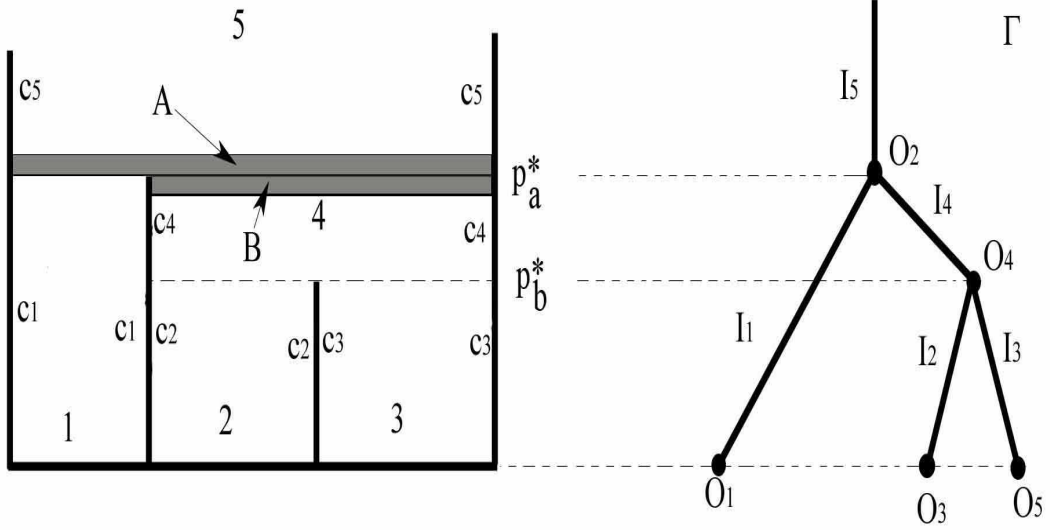


Fig. 6: The case when the system has more than two trapping wells

3 Regularization by stochastic perturbation of the dynamics

Let us now consider small stochastic perturbation of the dynamics rather than the initial condition. We will show that, this regularization works when the system has any number of trapping wells. But let us first start from the case of system with only two trapping wells.

The particle of unit mass starts its motion from $x = (q_0, p_0)$ with $p_0 > 0$ (see Fig.7). Each time when the particle hits the wall $q = -a_1$ or $q = a_2$, it instantaneously moves to $(-a_1, -p(1 - \varepsilon c_1 - \varepsilon \delta \eta_k))$, $p < 0$ and $(a_2, -p(1 - \varepsilon c_2 - \varepsilon \delta \xi_k))$, $p > 0$, respectively, and then it is reflected. We assume for simplicity that c_1, c_2 are positive constants, $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$. The restitution coefficient for the interior wall is assumed to be $1 - \varepsilon c_3 - \varepsilon \delta \zeta_k$ for both faces. And c_3 is also a positive constant.

Here ξ_k, η_k, ζ_k , $k = 1, 2, \dots$, are independent sequences of i.i.d random variables with continuous densities; the random variable with subscript k is used in the restitution coefficient when the trajectory hits the corresponding wall k -th time. In the following we also use the general notation ξ, η, ζ to denote independent r.v.'s which has the same distribution as ξ_k, η_k and ζ_k , respectively. We assume for brevity that random variables ξ, η and ζ are bounded with probability 1. Then, without loss of generality, one can consider just strictly positive ξ, η and ζ : $P\{\alpha < \xi < \beta\} = P\{\alpha < \eta < \beta\} = P\{\alpha < \zeta < \beta\} = 1$ for some $0 < \alpha < \beta < \infty$. Actually, one can replace the boundedness of

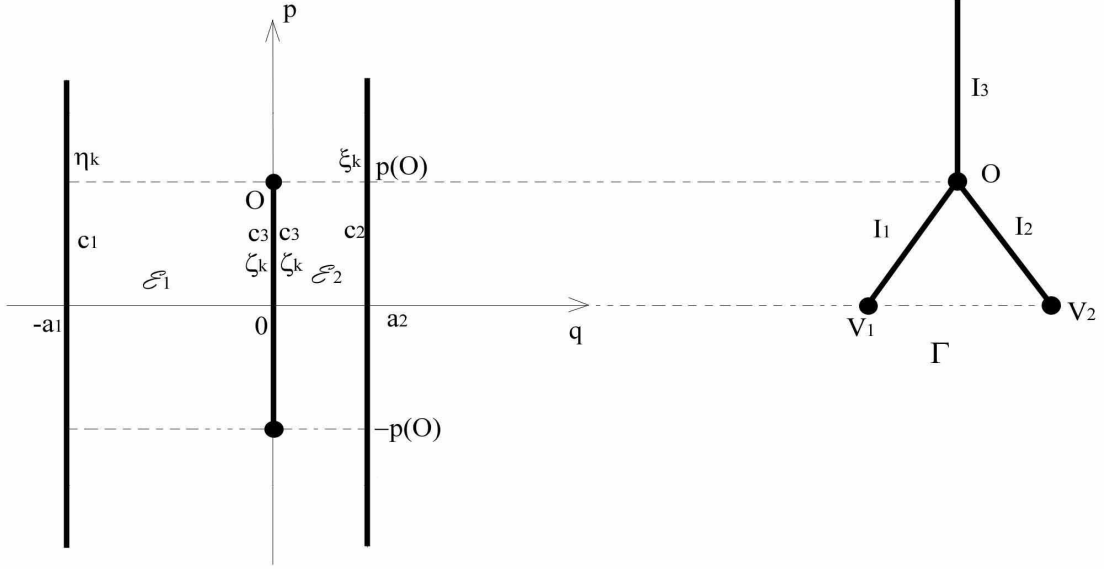


Fig. 7: Regularization by stochastic perturbation of the dynamics

random variables by their positivity and the finiteness of their moments (together with the existence of a continuous density).

The resulting process is denoted by $X_t^{\varepsilon, \delta} = (q^{\varepsilon, \delta}(t), p^{\varepsilon, \delta}(t))$. We rescale time and put $\tilde{X}_t^{\varepsilon, \delta} = X_{t/\varepsilon}^{\varepsilon, \delta}$. Let $Y_t^{\varepsilon, \delta} = (H(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$ be the projection of $\tilde{X}_t^{\varepsilon, \delta}$ onto the graph Γ (see Fig.7). Put $H^{\varepsilon, \delta}(t) = H(\tilde{X}_t^{\varepsilon, \delta})$. For fixed $\varepsilon > 0$, this is a stochastic process with jumps and each sample path is a step function which is right continuous and has limit from the left. Using the same construction as in the end of Section 2, we obtain from $H^{\varepsilon, \delta}(t)$ a continuous piecewise linear process $\hat{H}^{\varepsilon, \delta}(t)$. We also put $\hat{Y}_t^{\varepsilon, \delta} = (\hat{H}(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$. Since ξ and η are bounded,

$$P(|\hat{H}^{\varepsilon, \delta}(t) - H^{\varepsilon, \delta}(t)| < C\varepsilon) = 1$$

for some constant $C > 0$.

Now we are going to prove the weak convergence of $\hat{Y}_t^{\varepsilon, \delta}$ in the space $C_{0T}(\Gamma)$ of continuous functions $[0, T] \rightarrow \Gamma$ provided with uniform topology, as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, to a stochastic process Y_t on the graph Γ . We do this through a series of lemmas.

Lemma 3.1 *For fixed $\delta > 0$, the family $\{\hat{Y}_t^{\varepsilon, \delta}(t)\}_{\varepsilon > 0}$ is tight in the space $C_{0T}(\Gamma)$.*

Proof: We have to verify the following (see [7], Ch.6, Theorem 6.4.2): there exist $\alpha, \beta > 0$, s.t. for any $h > 0$, any $0 < t < t + h < T$, any $\varepsilon > 0$,

$$\mathbb{E} \left(\rho(\hat{Y}_t^{\varepsilon, \delta}, \hat{Y}_{t+h}^{\varepsilon, \delta}) \right)^\alpha \leq Mh^{1+\beta}.$$

Here $\rho(x, y)$ denotes the distance between the points $x, y \in \Gamma$ defined as follows: if x and y lies in the same edge I_i ($i = 1, 2, 3$) of the graph Γ , then $\rho(x, y)$ is just the usual Euclidean distance between points x and y within the line segment; otherwise, suppose $x \in I_i$ and $y \in I_j$, $i \neq j$, then consider a path lying on the graph Γ connecting x and y , we denote $\rho(x, y)$ be just the minimal length of such paths. Since our c_1, c_2, c_3 , and ξ, η, ζ are bounded with probability 1, and the period $T(H)$ of the non-perturbed (elastic) motion is bounded for $0 < H < \infty$ and separated from zero, we see that $P(\rho(\hat{Y}_t^\varepsilon, \hat{Y}_{t+h}^\varepsilon) < Mh) = 1$ for some constant $M > 0$. Therefore one can take $\alpha = 2, \beta = 1$, and the statement follows. ■

Lemma 3.2 *Let $H^{\varepsilon, \delta}(0) = H_0 > H(O)$. Within each edge of the graph Γ , as $\varepsilon \downarrow 0$, the process $\hat{H}^{\varepsilon, \delta}(t)$, for $0 < t < T < \infty$, converges uniformly in probability to a deterministic motion $H^\delta(t)$ which is defined by the equations*

$$H^\delta(t) = \left(\frac{t}{\sqrt{2}} \frac{c_1 + c_2 + \delta(\mathbb{E}\xi + \mathbb{E}\eta)}{a_1 + a_2} + H_0^{-1/2} \right)^{-2}, \text{ on } I_3,$$

and

$$H^\delta(t) = \left(\frac{t - t_0}{\sqrt{2}} \frac{c_1 + c_3 + \delta(\mathbb{E}\zeta + \mathbb{E}\eta)}{a_1} + H(O)^{-1/2} \right)^{-2}, \text{ on } I_1,$$

$$H^\delta(t) = \left(\frac{t - t_0}{\sqrt{2}} \frac{c_2 + c_3 + \delta(\mathbb{E}\xi + \mathbb{E}\zeta)}{a_2} + H(O)^{-1/2} \right)^{-2}, \text{ on } I_2,$$

respectively. Here $H(O)$ is the energy corresponding to the interior vertex O and $t_0^\delta = \frac{\sqrt{2}(a_1 + a_2)(H(O)^{-1/2} - H_0^{-1/2})}{c_1 + c_2 + \delta\mathbb{E}(\xi + \eta)}$ is the time for the motion $H^\delta(t)$ to come to the interior vertex O .

Proof: The proof of this lemma is similar to the proof of Lemma 2.1 and we use the same notations. The system (2.4) should be replaced by the following system:

$$\begin{cases} \dot{Q}_{t/\varepsilon}^\varepsilon = \frac{1}{\varepsilon} \sqrt{2H_{t/\varepsilon}^\varepsilon}, \\ \dot{H}_{t/\varepsilon}^\varepsilon = -[\Delta(Q_{t/\varepsilon}^\varepsilon - a_2) + \Delta(Q_{t/\varepsilon}^\varepsilon + a_1)](2c(Q_{t/\varepsilon}^\varepsilon) + 2\delta \cdot r(Q_{t/\varepsilon}^\varepsilon) - \varepsilon b(Q_{t/\varepsilon}^\varepsilon))H_{t/\varepsilon}^\varepsilon. \end{cases} \quad (3.1)$$

Here $\Delta(\cdot)$ is the Dirac δ -function so that the right hand side of the last equation in (3.1) is not zero just if $Q_{t/\varepsilon}^\varepsilon = a_1$ or $Q_{t/\varepsilon}^\varepsilon = a_2$; $r(Q_{t/\varepsilon}^\varepsilon) = \xi_k$ and $c(Q_{t/\varepsilon}^\varepsilon) = c_2$ when $c(Q_{t/\varepsilon}^\varepsilon) = a_2$; $r(Q_{t/\varepsilon}^\varepsilon) = \eta_k$ and $c(Q_{t/\varepsilon}^\varepsilon) = c_1$ when $c(Q_{t/\varepsilon}^\varepsilon) = a_1$. The function $b(Q_{t/\varepsilon}^\varepsilon) = \left(c(Q_{t/\varepsilon}^\varepsilon) + \delta \cdot r(Q_{t/\varepsilon}^\varepsilon) \right)^2$ is uniformly bounded for $0 \leq t \leq T$ with probability one. After that, using the same arguments as in Lemma 2.1 and the law of large numbers,

we obtain an equation for the limiting slow component $H(t)$ on each edge. For constant c_i , these equations can be solved explicitly and we get the statement of Lemma 3.2. ■

Lemma 3.2 implies the following:

Corollary 3.1 *Within each edge of the graph Γ , for the time $0 < t < T < \infty$, the process $\widehat{H}^{\varepsilon, \delta}(t)$, as $\varepsilon, \delta \downarrow 0$, converges in probability to a deterministic motion $H(t)$ defined by the equations*

$$H(t) = \left(\frac{t}{\sqrt{2}} \frac{c_1 + c_2}{a_1 + a_2} + H_0^{-1/2} \right)^{-2}, \text{ on } I_3 \quad (3.2)$$

and

$$H(t) = \left(\frac{t - t_0}{\sqrt{2}} \frac{c_1 + c_3}{a_1} + H(O)^{-1/2} \right)^{-2}, \text{ on } I_1 \quad (3.3)$$

$$H(t) = \left(\frac{t - t_0}{\sqrt{2}} \frac{c_2 + c_3}{a_2} + H(O)^{-1/2} \right)^{-2}, \text{ on } I_2 \quad (3.4)$$

respectively. Here $t_0 = \frac{\sqrt{2}(a_1 + a_2)(H(O)^{-1/2} - H_0^{-1/2})}{c_1 + c_2}$.

Let us consider now the slow motion near the interior vertex O . We first prove the following auxiliary lemma concerning random walks. Let $\{\xi_k\}$ and $\{\eta_k\}$ be independent sequences of i.i.d random variables. Assume that the random variables have continuous densities and $P\{\alpha < \xi < \beta\} = P\{\alpha < \eta < \beta\} = 1$ for some $0 < \alpha < \beta < \infty$. Put

$$S_0^x = x, \quad S_{2m}^x = x + \sum_{k=1}^m (\xi_k + \eta_k), \quad S_{2m+1}^x = x + S_{2m}^x + \xi_{m+1}.$$

It is clear that $S_n^x = x + S_n^0$, $P\{n\alpha < S_n^0 < n\beta\} = 1$.

Define $\tau_n^{x, \lambda}$ as the first time m when S_m^x is greater than $n\lambda$: $\tau_n^{x, \lambda} = \min\{m : S_m^x > n\lambda\}$.

Since $\mathbb{E}(\xi_k + \eta_k) > 0$, the law of large numbers implies that $P\{\tau_n^{x, \lambda} < \infty\} = 1$ for any $x \in \mathbb{R}^1$, $\lambda > 0$, $n \in \mathbb{Z}$.

We use two equivalent types of notations: The initial point x of the random walk will be included either as a superscript, like S_n^x , $\tau_n^{x, \lambda}$, or as a subscript in probabilities and expected values so that $\mathbb{E}_x f(S_n) \equiv \mathbb{E} f(S_n^x)$, $P_x\{\tau_n^\lambda < t\} \equiv P\{\tau_n^{x, \lambda} < t\}$.

Lemma 3.3 *Under the conditions mentioned above, $\lim_{n \rightarrow \infty} P_0\{\tau_n^\lambda \text{ is even}\} = \frac{\mathbb{E}\eta}{\mathbb{E}\xi + \mathbb{E}\eta}$, $\lim_{n \rightarrow \infty} P_0\{\tau_n^\lambda \text{ is odd}\} = \frac{\mathbb{E}\xi}{\mathbb{E}\xi + \mathbb{E}\eta}$.*

Proof: Put $m_1(n) = m_1 = \left\lceil \frac{n\lambda}{2\beta} \right\rceil$ if the latter integer is even, and $m_1(n) = m_1 = \left\lceil \frac{n\lambda}{2\beta} \right\rceil - 1$ otherwise. It is clear that $S_{m_1}^0 < \frac{n\lambda}{2}$. Put $M = \mathbb{E}(\xi_k + \eta_k)$, $D = \text{Var}(\xi_k + \eta_k)$. Let $\widetilde{m}_1(n) = \widetilde{m}_1 = \frac{m_1}{2}$. Let $f_{m_1}(x)$ be the density of $S_{m_1}^0 - \widetilde{m}_1 M$.

It follows from the central limit theorem that for each $\delta > 0$ one can choose $N = N(\delta)$ so that

$$P\{-N\sqrt{\widetilde{m}_1(n)} < S_{m_1}^0 - \widetilde{m}_1 M < N\sqrt{\widetilde{m}_1(n)}\} > 1 - \delta \quad (3.5)$$

for all large enough n .

Using the Markov property of the random walk S_n^x , we get

$$P_0\{\tau_n^\lambda \text{ is even}\} = \mathbb{E}_0 P_{S_{m_1(n)}}\{\tau_n^\lambda \text{ is even}\}. \quad (3.6)$$

One can conclude from (3.5) and (3.6) that

$$\left| P_0\{\tau_n^\lambda \text{ is even}\} - \int_{-N\sqrt{\widetilde{m}_1}}^{N\sqrt{\widetilde{m}_1}} f_{m_1}(x) P_{x+\widetilde{m}_1 M}\{\tau_n^\lambda \text{ is even}\} dx \right| < \delta. \quad (3.7)$$

Since our random variables are bounded and have density, one can apply the local central limit theorem to $S_{m_1(n)}^0$:

$$\left| f_{m_1}(x) - \frac{1}{\sqrt{2\pi\widetilde{m}_1 D}} \exp\left(-\frac{x^2}{2\widetilde{m}_1 D}\right) \right| < \frac{\rho_{m_1}^{(1)}}{\sqrt{\widetilde{m}_1}} \quad (3.8)$$

uniformly in $x \in \mathbb{R}^1$; here and later, we denote by $\rho_\ell^{(k)}$ such sequences that $\lim_{\ell \rightarrow \infty} \rho_\ell^{(k)} = 0$.

Divide the interval $[-N\sqrt{\widetilde{m}_1(n)}, N\sqrt{\widetilde{m}_1(n)}]$ into $\left\lceil \sqrt[4]{\widetilde{m}_1(n)} \right\rceil$ equal intervals I_1, \dots, I_r . It is clear that $r = r(n) \sim n^{1/4}$ and the length $|I_k|$ of each interval I_k is of order $n^{1/4}$ as $n \rightarrow \infty$. Because of (3.7) and (3.8),

$$\left| P_0\{\tau_n^\lambda \text{ is even}\} - \sum_{k=1}^{r(n)} \int_{I_k} \exp\left(-\frac{x^2}{2\widetilde{m}_1 D}\right) \frac{P_{x+\widetilde{m}_1 M}\{\tau_n^\lambda \text{ is even}\} dx}{\sqrt{2\pi\widetilde{m}_1 D}} \right| < \delta + \rho_n^{(2)}. \quad (3.9)$$

On each I_k , $k \in \{1, \dots, r\}$, choose a point z_k such that

$$\int_{I_k} \exp\left(-\frac{x^2}{2\widetilde{m}_1 D}\right) dx = |I_k| \exp\left(-\frac{z_k^2}{2\widetilde{m}_1 D}\right).$$

If $x \in I_k$, then

$$\left| \exp\left(-\frac{x^2}{2\widetilde{m}_1 D}\right) - \exp\left(-\frac{z_k^2}{2\widetilde{m}_1 D}\right) \right| \leq \rho_n^{(3)} \exp\left(-\frac{z_k^2}{2\widetilde{m}_1 D}\right), \quad (3.10)$$

where $\lim_{n \rightarrow \infty} \rho_n^{(3)} = 0$ since $|x| < N\sqrt{\widetilde{m}_1(n)}$, $|z_k| < N\sqrt{\widetilde{m}_1(n)}$, $|x - z_k| < \text{Const} \cdot n^{1/4}$, $\widetilde{m}_1(n)$ is of order n as $n \rightarrow \infty$. We conclude from (3.9) and (3.10):

$$\left| P_0\{\tau_n^\lambda \text{ is even}\} - \sum_{k=1}^{r(n)} \exp\left(-\frac{z_k^2}{2\widetilde{m}_1 D}\right) \frac{|I_k|}{\sqrt{2\pi\widetilde{m}_1 D}} \int_{I_k} P_{x+\widetilde{m}_1 M}\{\tau_n^\lambda \text{ is even}\} \frac{dx}{|I_k|} \right| \leq \delta + \rho_n^{(4)}. \quad (3.11)$$

Note that, since the random walk S_n is invariant with respect to shifts,

$$P_{x+An}\{\tau_n^\lambda \text{ is even}\} = P_x\{\tau_n^{\lambda+A} \text{ is even}\}.$$

Using this fact and taking into account (3.8), we conclude that

$$\int_{I_k} \frac{dx}{|I_k|} P_{x+\widetilde{m}_1 M}\{\tau_n^\lambda \text{ is even}\} = \int_{I_k} \frac{dx}{|I_k|} P_x\{\tau_n^{\lambda'} \text{ is even}\} + \frac{\rho_n^{(5)}}{\sqrt{n}} \quad (3.12)$$

for an appropriate constant λ' . The integral in the right hand side of (3.12) is nothing else but the expected value $\widetilde{\mathbb{E}}$ of the random variable X uniformly distributed on I_k . Let $\chi_x^{(n)}$ be the indicator of the set $\{\tau_n^{\lambda'} \text{ is even}\}$ in the sample space. Then

$$\int_{I_k} \frac{dx}{|I_k|} P_x\{\tau_n^{\lambda'} \text{ is even}\} = \widetilde{\mathbb{E}}\mathbb{E}\chi_X^{(n)} = \mathbb{E}\widetilde{\mathbb{E}}\chi_X^{(n)}. \quad (3.13)$$

For given sequences $\{\xi_k\}$, $\{\eta_k\}$,

$$\widetilde{\mathbb{E}}\chi_X^{(n)} = \frac{\eta_1 + \eta_2 + \dots + \eta_{\nu_n} + \kappa_n}{\xi_1 + \eta_1 + \xi_2 + \eta_2 + \dots + \xi_{\nu_n} + \eta_{\nu_n}}, \quad (3.14)$$

where ν_n and κ_n are random variables such that $\lim_{n \rightarrow \infty} \nu_n = \infty$ and $|\kappa_n| < \beta$.

Using the strong law of large numbers, we conclude from (3.14) that

$$\lim_{n \rightarrow \infty} \widetilde{\mathbb{E}}\chi_X^n = \frac{\mathbb{E}\eta}{\mathbb{E}\xi + \mathbb{E}\eta}, \quad a.e. .$$

Since $|\widetilde{\mathbb{E}}\chi_X^{(n)}| \leq 1$, the last equality and (3.13) imply that

$$\lim_{n \rightarrow \infty} \int_{I_k} \frac{dx}{|I_k|} P_x\{\tau_n^{\lambda'} \text{ is even}\} = \frac{\mathbb{E}\eta}{\mathbb{E}\xi + \mathbb{E}\eta}. \quad (3.15)$$

From (3.11), (3.12) and (3.15), we derive:

$$\left| P_0\{\tau_n^\lambda \text{ is even}\} - \frac{\mathbb{E}\eta}{\mathbb{E}\xi + \mathbb{E}\eta} \right| \leq 2\delta + \rho_n^{(6)}.$$

This bound implies the first statement of the lemma. The second statement follows from the first. ■

Now we have, for our process $\tilde{X}_t^{\varepsilon, \delta}$,

Lemma 3.4

$$\begin{aligned} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\tilde{X}_t^{\varepsilon, \delta} \text{ finally falls into the well } \mathcal{E}_1\} &= \frac{c_1}{c_1 + c_2}, \\ \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\tilde{X}_t^{\varepsilon, \delta} \text{ finally falls into the well } \mathcal{E}_2\} &= \frac{c_2}{c_1 + c_2}. \end{aligned}$$

(see Fig.7)

Proof: When the particle collides with the wall $q = -a_1$, the absolute value of its velocity changes as follows:

$$|p_{new}| = |p_{old}|(1 - \varepsilon c_1 - \varepsilon \delta \xi_k).$$

Similarly, when the particle collides with the wall $q = a_2$:

$$|p_{new}| = |p_{old}|(1 - \varepsilon c_2 - \varepsilon \delta \eta_k).$$

Take logarithm of these two equalities:

$$\ln |p_{new}| = \ln |p_{old}| + \ln(1 - \varepsilon c_1 - \varepsilon \delta \xi_k),$$

$$\ln |p_{new}| = \ln |p_{old}| + \ln(1 - \varepsilon c_2 - \varepsilon \delta \eta_k),$$

respectively. Thus we can view the problem of determining whether $\tilde{X}_t^{\varepsilon, \delta}$ enters well \mathcal{E}_1 or \mathcal{E}_2 as the following random walk problem. We start from $\ln |p_0|$, and alternatively jump backward with steplength $-\ln(1 - \varepsilon c_1 - \varepsilon \delta \xi_k)$ (at step $2k - 1$) and $-\ln(1 - \varepsilon c_2 - \varepsilon \delta \eta_k)$ (at step $2k$). If the random walk finally jumps over $\ln |p(O)|$ at an odd step, then $\tilde{X}_t^{\varepsilon, \delta}$ enters well \mathcal{E}_1 ; otherwise $\tilde{X}_t^{\varepsilon, \delta}$ enters well \mathcal{E}_2 .

Now we further simplify the problem: we start from 0, and alternatively jump forward with steplength $U_k^{\varepsilon, \delta} = -\frac{1}{\varepsilon} \ln(1 - \varepsilon c_1 - \varepsilon \delta \xi_k)$ (at step $2k - 1$) and $V_k^{\varepsilon, \delta} = -\frac{1}{\varepsilon} \ln(1 - \varepsilon c_2 - \varepsilon \delta \eta_k)$ (at step $2k$). If the random walk finally jumps over $\frac{1}{\varepsilon} \ln \left| \frac{p_0}{p(O)} \right|$ at an odd step, then $\tilde{X}_t^{\varepsilon, \delta}$ enters well \mathcal{E}_1 ; otherwise $\tilde{X}_t^{\varepsilon, \delta}$ enters well \mathcal{E}_2 .

Put $n = \left\lceil \frac{1}{\varepsilon} \right\rceil$, $\lambda = \ln \left| \frac{p_0}{p(O)} \right|$. Taking into account that $U_k^{\varepsilon, \delta} = (c_1 + \delta \xi_k) + O(\varepsilon)$ and $V_k^{\varepsilon, \delta} = (c_2 + \delta \eta_k) + O(\varepsilon)$ as $\varepsilon \downarrow 0$ and applying Lemma 3.3 we get:

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\tilde{X}_t^{\varepsilon, \delta} \text{ finally falls into the well } \mathcal{E}_1\} = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}U_k^{\varepsilon, \delta}}{\mathbb{E}U_k^{\varepsilon, \delta} + \mathbb{E}V_k^{\varepsilon, \delta}} = \frac{c_1}{c_1 + c_2}.$$

The second statement of the lemma follows from the first. ■

Let $Y_t = (H_t, K_t)$, $H_0 > H(O)$, $K_0 = 3$, be the process on Γ defined inside the edges by formulas given in Corollary (3.1); when Y_t reaches at time $t_0 = \frac{\sqrt{2}(a_1 + a_2)}{c_1 + c_2}(H(O)^{-1/2} - H_0^{-1/2})$ the interior vertex O , it goes immediately to I_1 or I_2 with probabilities $p_1 = \frac{c_1}{c_1 + c_2}$ and $p_2 = \frac{c_2}{c_1 + c_2}$ respectively.

Combining Lemma 3.1 and Corollary 3.1 and Lemma 3.4 we have the following

Theorem 3.1 *The process $\hat{Y}_t^{\varepsilon, \delta} = (\hat{H}(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$ on Γ converges weakly in uniform topology in the space of continuous functions $[0, T] \rightarrow \Gamma$ as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ to the stochastic process $Y_t = (H(t), K(t))$ on Γ .*

Remark: As we have seen in Section 2, in the case of two wells, the problem can be regularized by stochastic perturbations of the initial conditions. Theorem 3.1 shows that the regularization by perturbations of the dynamics lead to the same limiting slow motion Y_t on Γ .

Now we consider the case when the model system has more than two trapping wells (see Fig.8). Then the regularization by perturbation of the initial conditions, in general, does not work. Let the coefficients of restitution of the left and right boundaries of the i -th well be $1 - \varepsilon c_1^{(i)} - \varepsilon \delta \xi_k^{(i)}$ for the left boundary and $1 - \varepsilon c_2^{(i)} - \varepsilon \delta \eta_k^{(i)}$ for the right boundary. In Fig.8 $i \in \{1, 2, 3, 4, 5\}$. A random variable with the subscript k is used when the particle hits a wall the k -th time; $c_1^{(i)}, c_2^{(i)}$ are positive constants; $\{\xi_k^{(i)}\}_{k=1}^\infty, \{\eta_k^{(i)}\}_{k=1}^\infty$ are sequences of i.i.d random variables. We write $\xi^{(i)}, \eta^{(i)}$ to denote independent random variables that have the same distributions as any of the random variables from the sequences $\{\xi_k^{(i)}\}_{k=1}^\infty, \{\eta_k^{(i)}\}_{k=1}^\infty$. We assume that $P\{\alpha < \xi^{(i)} < \beta\} = P\{\alpha < \eta^{(i)} < \beta\} = 1$ for $0 < \alpha < \beta < \infty$.

The resulting process is denoted by $X_t^{\varepsilon, \delta} = (q^{\varepsilon, \delta}(t), p^{\varepsilon, \delta}(t))$. We rescale time and put $\tilde{X}_t^{\varepsilon, \delta} = X_{t/\varepsilon}^{\varepsilon, \delta}$. Let $Y_t^{\varepsilon, \delta} = (H(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$ be the projection of $\tilde{X}_t^{\varepsilon, \delta}$ onto the graph Γ (see Fig.8). Put $H^{\varepsilon, \delta}(t) = H(\tilde{X}_t^{\varepsilon, \delta})$. For fixed $\varepsilon > 0$, this is a stochastic process with jumps and each sample path is a step function which is right continuous and has limit from the left. Using the same construction as in the end of Section 2, we obtain from $H^{\varepsilon, \delta}(t)$ a continuous piecewise linear process $\hat{H}^{\varepsilon, \delta}(t)$. We also put $\hat{Y}_t^{\varepsilon, \delta} = (\hat{H}(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$.

The averaging procedure is the same as before. One can see that within each edge I_i , as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, the process $H^{\varepsilon, \delta}(t)$ converges to a deterministic motion

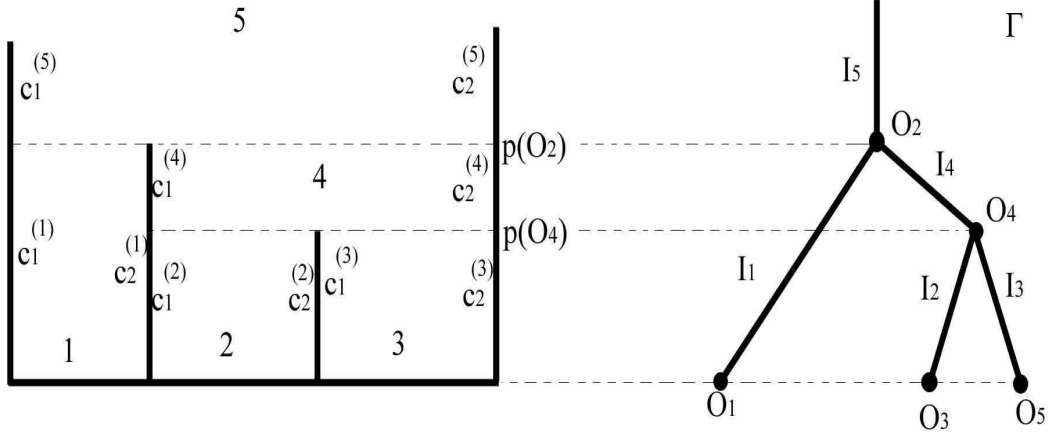


Fig. 8: The case when the system has more than two trapping wells

$H(t)$ which satisfies the differential equation

$$\frac{dH}{dt} = -2 \frac{c_1^{(i)} + c_2^{(i)}}{T_i(H)} H, \quad (3.16)$$

where $i \in \{1, 2, 3, 4, 5\}$. $T_i(H)$ is the period of motion in the phase space for the particle in the i -th well with energy H : $T_i(H) = \frac{\sqrt{2}m_i}{\sqrt{H}}$ where m_i is the width of the i -th well.

The branching probabilities for the limiting motion $Y(t) = (H(t), K(t))$ at each interior vertex O_l are given by $p_1^{(l)} = \frac{c_1^{(i)}}{c_1^{(i)} + c_2^{(i)}}$ and $p_2^{(l)} = \frac{c_2^{(i)}}{c_1^{(i)} + c_2^{(i)}}$. Here $i = i(l)$ is the number of the edge that is above O_l ; $p_1^{(l)}$, $p_2^{(l)}$ are probabilities that the particle enters the left edge or right edge that is below O_l . The motion inside the edges is described as in Corollary (3.1). Since each time when the particle hits the boundary, the perturbation of the dynamics is given by independent random variables, the branching at each interior vertex is also independent of each other. Therefore we finally have the following theorem:

Theorem 3.2 *The process $\hat{Y}_t^{\varepsilon, \delta} = (\hat{H}(\tilde{X}_t^{\varepsilon, \delta}), K(\tilde{X}_t^{\varepsilon, \delta}))$ on Γ converges weakly in uniform topology in the space of continuous functions $[0, T] \rightarrow \Gamma$ as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ to the continuous stochastic process $Y_t = (H(t), K(t))$ on Γ . The process Y_t , $Y_0 = Y(q_0, p_0)$, inside the edges is described by (3.16). When the process Y_t along an edge I_i attached to an interior vertex O_l and situated above O_l reaches O_l , it instantaneously leaves O_l and enters edges of Γ attached to O_l and situated below O_l on the left and on the right with probabilities $p_1^{(l)} = \frac{c_1^{(i)}}{c_1^{(i)} + c_2^{(i)}}$, $p_2^{(l)} = \frac{c_2^{(i)}}{c_1^{(i)} + c_2^{(i)}}$, respectively. The branching*

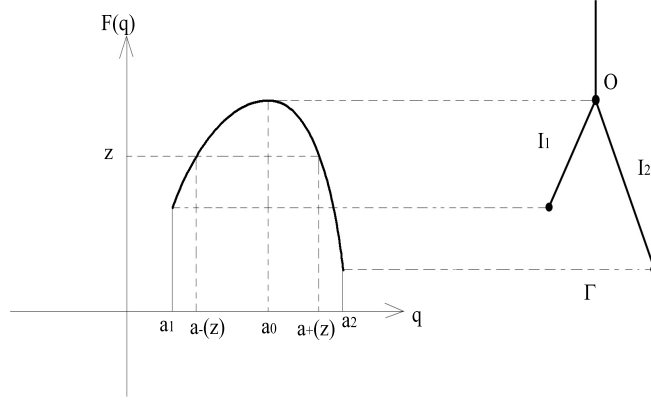


Fig. 9: The case of general potential

at each interior vertex O_l is independent of each other.

Remark: Let us suppose that $\delta = 1$: the coefficients of restitution become $1 - \varepsilon c_1^{(i)} - \varepsilon \xi_k^{(i)}$ and $1 - \varepsilon c_2^{(i)} - \varepsilon \eta_k^{(i)}$, respectively. Then the process $\tilde{Y}_t^{\varepsilon,1}$ converges weakly as $\varepsilon \downarrow 0$ to the process \tilde{Y}_t on Γ described above with the replacement of $c_1^{(i)}$ and $c_2^{(i)}$ by $\tilde{c}_1^{(i)} = c_1^{(i)} + \mathbb{E}\xi^{(i)}$ and by $\tilde{c}_2^{(i)} = c_2^{(i)} + \mathbb{E}\eta^{(i)}$, respectively.

4 The case of general potential

The problem mentioned in the beginning of the introduction is considered in this section. Let the motion of a particle inside $[a_1, a_2]$ be governed by the equation $\ddot{q}_t^\varepsilon = -F'(q_t^\varepsilon)$, $q_0^\varepsilon = q_0$, $\dot{q}_0^\varepsilon = p_0$. When the particle hits the walls, it is reflected with a loss of energy. Let $H(p, q) = \frac{p^2}{2} + F(q)$ be the energy of the particle, $p = \dot{q}$. The coefficients of restitution at a_i are equal to $1 - \varepsilon c_i(H)$, $0 < \varepsilon \ll 1$, $i \in \{1, 2\}$, $c_i(H)$ are positive and smooth. As was explained in the introduction, we can restrict ourselves to the case when $F(q)$ has just one maximum inside $[a_1, a_2]$, say, at $a_0 \in (a_1, a_2)$ (see Fig.9). Let Γ be the graph corresponding to the Hamiltonian $H(p, q)$, $\square = \{(p, q) \in \mathbb{R}^2 : a_1 \leq q \leq a_2\}$, and $Y : \square \rightarrow \Gamma$ be the projection of \square on Γ : $Y(p, q) = (H(p, q), K(p, q))$.

Denote by $X_t^\varepsilon = (q_t^\varepsilon, p_t^\varepsilon)$ nearly-elastic trajectory starting at (q_0, p_0) . This motion has a fast component, which is actually the motion along the elastic trajectories, and the slow component $Y_t^\varepsilon = Y(X_t^\varepsilon)$. Let $\tilde{Y}_t^\varepsilon = Y_{t/\varepsilon}^\varepsilon$ and let $\hat{Y}_{t/\varepsilon}^\varepsilon$ be the continuous piecewise linear approximation of $Y_{t/\varepsilon}^\varepsilon$ as we considered earlier. As in the model problem, one can see that $\lim_{\varepsilon \downarrow 0} \hat{Y}_t^\varepsilon$ (and $\lim_{\varepsilon \downarrow 0} \tilde{Y}_t^\varepsilon$) does not exist. Since we have just two wells, one can

use perturbations of the initial conditions to regularize the problem. Let ξ_δ be a two-dimensional random variable with a continuous density which is positive for $|x| < \delta$ and is equal to zero if $|x| > \delta$. Denote by $\widehat{Y}_t^{\varepsilon, \delta}$ the continuous modification introduced above when the initial condition $x_0 = (q_0, p_0)$ is replaced by $x_0 + \xi_\delta$; $\widehat{Y}_t^{\varepsilon, \delta}$ is a stochastic process since $\widehat{Y}_0^{\varepsilon, \delta}$ is random.

Let us introduce now a stochastic process $Y_t = (H_t, K_t)$ on Γ . First, let $T_k(H)$, $k \in \{1, 2, 3\}$, be the period of oscillations of the elastic motion q_t^0 with the initial conditions in $Y^{-1}(H, K)$. It is easy to check that

$$\begin{aligned} T_3(H) &= 2 \int_{a_1}^{a_2} \frac{dq}{\sqrt{2(H - F(q))}} , \\ T_1(H) &= 2 \int_{a_1}^{a_-(H)} \frac{dq}{\sqrt{2(H - F(q))}} , \\ T_2(H) &= 2 \int_{a_+(H)}^{a_2} \frac{dq}{\sqrt{2(H - F(q))}} . \end{aligned}$$

Here $a_\pm(z)$ are roots of the equation $F(a_\pm(z)) = z$, $a_-(z) < a_+(z)$.

Let $Y_t = (H_t, K_t)$ be the stochastic process on Γ such that $H_0 = H(q_0, p_0) > H(O)$, $K_0 = 3$; H_t is a deterministic motion inside each edge:

$$\dot{H}_t = -2 \frac{c_k^{(1)}(H_t)(H_t - F(a_k^{(1)})) + c_k^{(2)}(H_t)(H_t - F(a_k^{(2)}))}{T_k(H_t)}$$

inside I_k , $k \in \{1, 2, 3\}$, where $c_k^{(j)}(H) = c_j(H)$, $a_k^{(j)} = a_j$, for $k = 3$ and $j = 1, 2$; $c_k^{(j)}(H) = c_k(H)$, $a_k^{(j)} = a_k$ for $k = 1, 2$ and $j = 1, 2$. The trajectory Y_t hits O in a finite time $t_0 = \int_{H(O)}^{H(q_0, p_0)} \frac{T_3(z) dz}{2[c_1(z)(z - F(a_1)) + c_2(z)(z - F(a_2))]}$ since $T_3(H) \sim \ln(H - H(O))$ as $H \rightarrow H(O)$. After hitting vertex O , Y_t leaves O immediately and goes to I_1 or I_2 with probabilities $p_1 = \frac{c_1(H(O))}{c_1(H(O)) + c_2(H(O))}$ and $p_2 = \frac{c_2(H(O))}{c_1(H(O)) + c_2(H(O))}$. These conditions define Y_t in a unique way.

Theorem 4.1 *Under mentioned above conditions, process $\widehat{Y}_t^{\varepsilon, \delta}$ converge weakly in the space C_{0T} of continuous functions $[0, T] \rightarrow \Gamma$, $T < \infty$, as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ to the process Y_t .*

The proof of this Theorem is similar to the proof of Theorem 2.1 and we omit it.

One can regularize the problem by introducing random perturbations not of the initial conditions, but of the dynamics, as we did in Section 3. Similar to Theorem 3.1, one can prove that the process Y_t introduced above again serves as the limiting slow

motion. These results allow to say that stochasticity is an intrinsic property of the long-time behavior of nearly-elastic systems: the limiting slow motion is the same stochastic process for different regularizations.

5 A two dimensional model problem

Consider a standard billiard model in which a point mass moves freely inside a convex, bounded, and simply connected region G in the plane with smooth boundary ∂G (Fig.10). The orbits of such motion consist of straight line segments inside G joined at boundary points according to the rule that the angle of incidence equals the angle of reflection. Speed is a constant of motion. The phase space Λ of this system is conveniently described as the set of all tangent vectors of fixed length (say, unit length) supported at points of the interior of G together with vectors at boundary points pointing inward. We parameterize the phase space Λ by the coordinate (g, φ) , where $g \in G \cup \partial G$ is the position of the particle within $G \cup \partial G$ and $\varphi \in [0, 2\pi)$ is the angle between the positive Y -axis and the velocity vector of the particle. The phase space of the motion has a global section Π which consists of all inward-pointing unit vectors on ∂G . Topologically Π is a cylinder parameterized by the cyclic length parameter $s \in (0, L]$ along ∂G (here L is the length of ∂G) and the angle $\theta \in [0, \pi]$ between the velocity and the positive tangential direction. The Poincaré map $f : \Pi \rightarrow \Pi$ is usually called *the billiard map* and can be described by $f(s, \theta) = (S(s, \theta), \Theta(s, \theta))$. Notice that when ∂G is smooth f is also smooth. For a further reference about the billiard model we refer to [9], page 339.

A classical result in such a billiard model is that f preserves the volume element $\sin \theta ds d\theta$ on Π . This volume element $\sin \theta ds d\theta$ is usually called *the Liouville measure*; $m(s, \theta) = \sin \theta$ is the density of the Liouville measure.

We now assume that the particle loses a small amount of energy each time when it has a collision with ∂G . Let $c(s, \theta)$ be a smooth function on Π , bounded from above and below by some fixed positive constants. We assume for brevity that the energy H of the particle after the collision with ∂G at a point $x \in \Pi$ diminishes to $\max\{H - \varepsilon c(x), 0\}$ (rather than to $H(1 - \varepsilon c(x))$ as before); here ε is a small positive parameter. The direction of the velocity vector remains the same as for the elastic system (see Fig.10).

We assume that there is another wall of height $H(O)$ separating the region G (see Fig.11). In a finite time of order $\frac{\text{const}}{\varepsilon}$, $\varepsilon \downarrow 0$, each trajectory of the nearly-elastic motion, which had at time zero energy $H_0 > H(O)$, enters one of the wells 1 or 2 shown in Fig.11 and continues the motion there (we assume that the loss of energy on the wall separating the region G is defined in a similar way). Which of the wells is entered, in general, depends on ε in a very sensitive way: as $\varepsilon \downarrow 0$ the trajectory alternatively enters well 1 or 2. As in the previous sections, the nearly-elastic motion has a fast

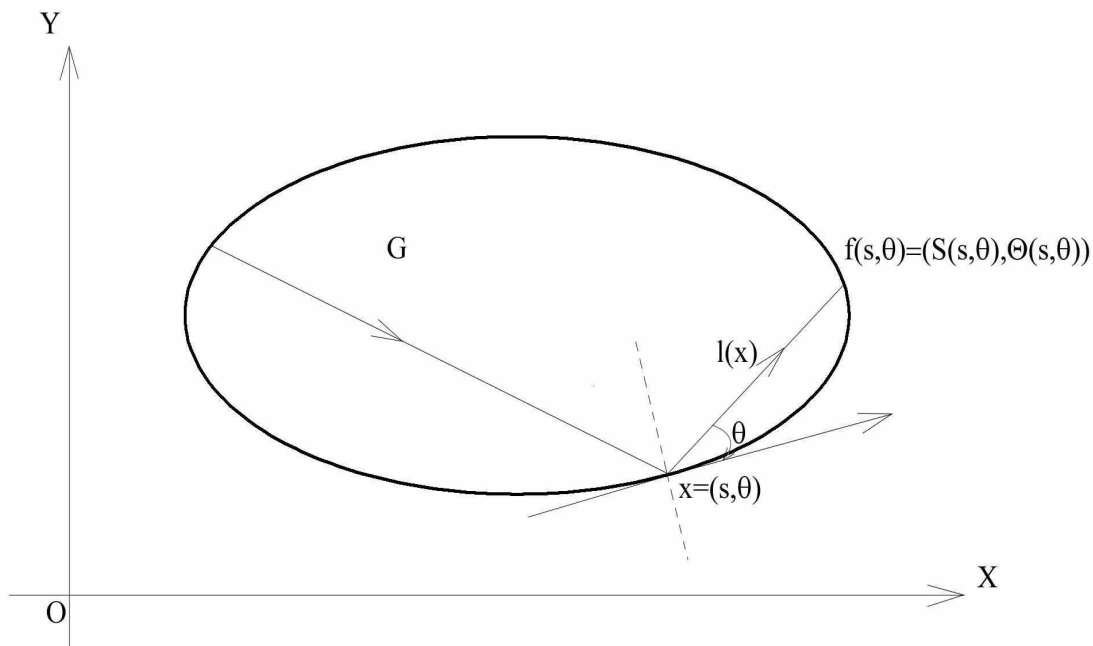


Fig. 10: The billiard model

and a slow components. In the case of one degree of freedom, the fast motion has just one normalized invariant measure, and a unique, independent of the initial conditions, limiting slow motion inside the edges exists without any regularization. This is not the case now. Besides the Liouville measure, other invariant measures on the Poincaré section Π exist; for instance, there are periodic points on ∂G . On the other hand, it is natural to assume that the reflections on ∂G are undergoing small random perturbations of intensity $\delta \ll 1$ so that the resulting motion is a stochastic process depending on two small parameters ε and δ . We choose a class of random perturbations in such a way that the perturbed system induces on Π a unique invariant measure, namely, the Liouville measure (compare with [5]). It turns out that the perturbations in the fast motion which we introduce provide a regularization of the slow component not just inside of the edges of the corresponding graph but also near the vertices, so that the weak limit of the slow component of the regularized motion (after an appropriate time change) exists on the whole graph as first ε and then δ tends to zero. Probably, the class of permissible random perturbations leading to the same limiting slow motion can be essentially extended. But one should keep in mind that the existence of a rich set of invariant measures for the billiard map shows that the limiting slow motion calculated in this section has, in a sense, a restricted universality: other regularizations of the fast motion may lead to different slow motions.

We regularize this problem by considering the following scheme of small stochastic

perturbation of the system. Consider an one-dimensional diffusion process I_t^θ on $[0, \pi]$ starting from point θ . The diffusion process is governed by a second order elliptic operator \mathcal{L} which is self-adjoint with respect to the Liouville measure. Such an operator can be written as $\mathcal{L}u = \frac{1}{2 \sin \theta} \frac{d}{d\theta} (a(\theta) \frac{du}{d\theta})$, where a is a smooth positive function of θ . We assume that the diffusion process I_t^θ has instantaneous reflection at the boundary points $\theta = 0, \pi$. Notice that there is a singularity of our operator \mathcal{L} at the boundary points $\theta = 0, \pi$. In spite of this singularity the boundary points $\theta = 0, \pi$ are accessible and therefore we need boundary conditions: we choose instantaneous reflection at these points.

Choose a small constant $\delta > 0$. Let the position of the particle after one collision be $x = (s, \theta) \in \Pi$ with an energy $H > 0$. Then the next position on Π will be not exactly x but $Z_\delta^x = (s, I_\delta^\theta)$, and the energy will diminish to $H - \varepsilon c(x)$ (see Fig.11). And the subsequent motion starts from this new point Z_δ^x . The same kind of perturbation is repeated again and again for each collision, independent of other collisions. This scheme of perturbation models the fact that each time when a collision happens, the particle changes its direction of motion a little bit, due to the random perturbations.

After adding the small stochastic perturbation, the evolution of energy and the motion of the particle is described by a continuous time Markov process $N_t^{\varepsilon, \delta} = (H_t^{\varepsilon, \delta}, \Lambda_t^{\varepsilon, \delta})$ on $[0, +\infty) \times \Lambda$. The energy $H_t^{\varepsilon, \delta}$ changes just on the boundary: if $\Lambda_t^{\varepsilon, \delta} = x = (s, \theta) \in \Pi$, the energy instantaneously decreases on $\varepsilon c(x)$; $H_t^{\varepsilon, \delta}$ is right continuous. The process $\Lambda_t^{\varepsilon, \delta} = (g_t^{\varepsilon, \delta}, \varphi_t^{\varepsilon, \delta})$ has a piecewise linear first component: it moves uniformly along the chords connecting successive points where $g_t^{\varepsilon, \delta}$ hits ∂G with the constant speed $\sqrt{2H_t^{\varepsilon, \delta}}$. Let $M_t^{\varepsilon, \delta} = (H_t^{\varepsilon, \delta}, g_t^{\varepsilon, \delta})$ (a sample trajectory of $M_t^{\varepsilon, \delta}$ is the thin line in Fig.11). The trajectory of $M_t^{\varepsilon, \delta}$ is right continuous and has limit from the left. We define a continuous modification $\widehat{M}_t^{\varepsilon, \delta} = (\widehat{H}_t^{\varepsilon, \delta}, g_t^{\varepsilon, \delta})$ of $M_t^{\varepsilon, \delta}$ by taking $\widehat{H}_t^{\varepsilon, \delta}$ as a piecewise linear modification of $H_t^{\varepsilon, \delta}$ (a sample trajectory of $\widehat{M}_t^{\varepsilon, \delta}$ is the bold line in Fig.11). Let $\widehat{N}_t^{\varepsilon, \delta} = (\widehat{H}_t^{\varepsilon, \delta}, \Lambda_t^{\varepsilon, \delta})$.

Consider also a jumping Markov chain $\{X_n^\delta\}_{n \geq 1}$ on Π , defined as follows: from a point $X_n^\delta = (s_n^\delta, \theta_n^\delta) \in \Pi$ the chain goes in one time unit to $X_{n+1}^\delta = (s_{n+1}^\delta, \theta_{n+1}^\delta) = (S(s_n^\delta, \theta_n^\delta), I_\delta^{\Theta(s_n^\delta, \theta_n^\delta)}) \in \Pi$.

Suppose the particle of unit mass starts its motion from a point $x = (s, \theta) \in \Pi$ and the energy of the particle is H . Let $T(x, H)$ be the time that the particle starting from x first reaches ∂G . Let $l(x)$ be the chord starting from $x \in \Pi$ (see Fig.10). Let $L(x)$ be the length of $l(x)$. We have $T(x, H) = \frac{L(x)}{\sqrt{2H}}$.

We identify points of each well having the same energy. The set obtained after such an identification is a graph Γ ; the metric on Γ is defined by the distance along the edges of Γ . This graph has one interior vertex O corresponding to the energy level $H(O)$ connected with three edges: I_3 corresponding to trajectories with energy greater than $H(O)$, I_1 corresponding to trajectories with energy less than $H(O)$ situated within well

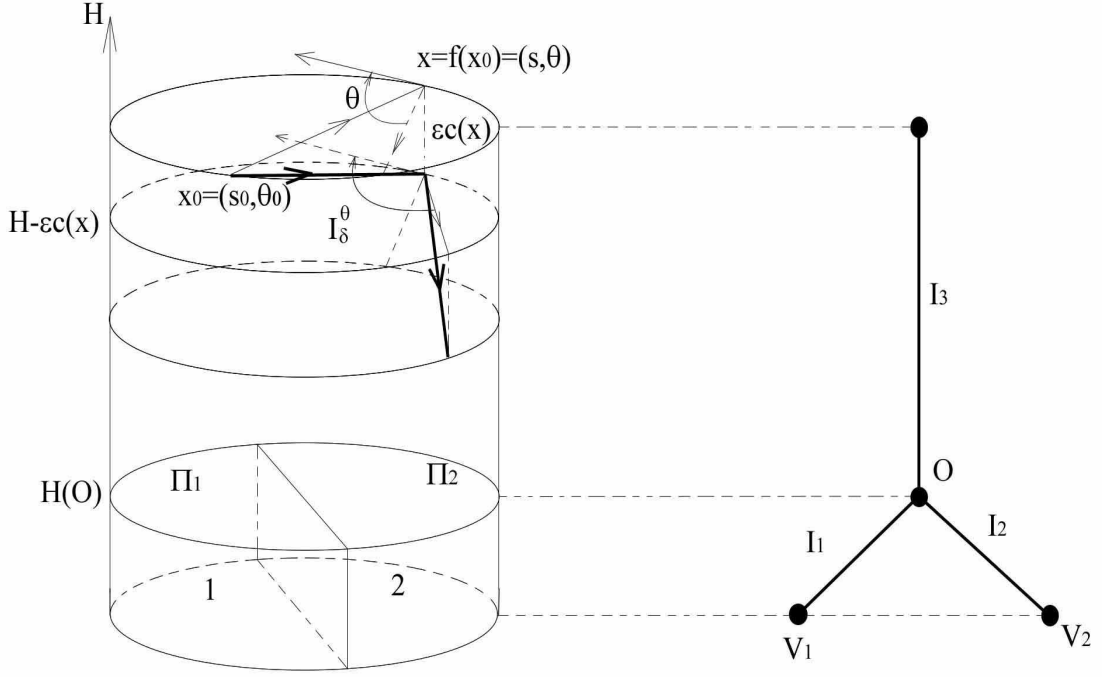


Fig. 11: The billiard model with energy loss and perturbation

1 (to the left of O), and I_2 corresponding to trajectories with energy less than $H(O)$ situated within well 2 (to the right of O).

We consider the process $Y_t^{\varepsilon, \delta} = (H_t^{\varepsilon, \delta}, K_t^{\varepsilon, \delta})$ on the tree Γ , where $K_t^{\varepsilon, \delta}$ is defined as follows: $K_t^{\varepsilon, \delta} = 3$ if the energy of the particle is greater than $H(O)$, $K_t^{\varepsilon, \delta} = 1$ if the particle has energy less than $H(O)$, while $N_t^{\varepsilon, \delta}$ is in the potential well 1, and $K_t^{\varepsilon, \delta} = 2$ if the particle has energy less than $H(O)$, while $N_t^{\varepsilon, \delta}$ is in the potential well 2. The process $Y_t^{\varepsilon, \delta}$ is the slow component of $N_t^{\varepsilon, \delta}$ as $\varepsilon \downarrow 0$. We show that the rescaled process $Y_{t/\varepsilon}^{\varepsilon, \delta}$ converges, as first $\varepsilon \downarrow 0$ then $\delta \downarrow 0$, to a stochastic process Y_t on the tree Γ . Such a stochastic process has deterministic motion within each edge of the tree Γ and has stochasticity only at the interior vertex O .

Let us first state some auxiliary results:

Lemma 5.1 *The diffusion process I_t^θ on $[0, \pi]$ governed by the operator $\mathcal{L}u = \frac{1}{2 \sin \theta} \frac{d}{d\theta} (a(\theta) \frac{du}{d\theta})$ with instantaneous reflection at boundary points $\theta = 0, \pi$, has $\frac{1}{2} \sin \theta$ as its unique invariant density.*

Suppose small $r > 0$, let $\Pi(r) = \{x = (s, \theta) \in \Pi; r \leq \theta \leq \pi - r\}$.

Lemma 5.2 *For any $\delta > 0$, for any small enough $r > 0$, the Markov chain*

$\{X_n^\delta\}_{n \geq 1}$ on $\Pi(r)$ satisfies the Doeblin condition, and has only one ergodic component on $\Pi(r)$. The invariant measure for the process $\{X_n^\delta\}_{n \geq 1}$ on Π has a density $d(x) = \frac{1}{2L}m(x) = \frac{1}{2L}\sin\theta$.

Lemma 5.3 Within edge I_3 of the graph Γ , $\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} H_{t/\varepsilon}^{\varepsilon, \delta} = H_t$ in probability, where H_t satisfies the following differential equation:

$$\frac{dH_t}{dt} = - \frac{\iint_{\Pi} c(x)m(x)dx}{\iint_{\Pi} T(x, H_t)m(x)dx}, \quad H_0 = H_0^{\varepsilon, \delta} \quad (5.1)$$

In exactly the same way as this lemma one can have similar results for limiting deterministic motion within edge I_1 and I_2 . We omit details here.

Lemma 5.4 Denote the area of the region bounded by the convex curve in Fig.10 by A . Then

$$\iint_{\Pi} L(x)m(x)dx = 2\pi A$$

where $m(x)$ is the density of the Liouville measure on Π and $L(x)$ is the length of the chord $l(x)$.

Using Lemma 5.4, equation (5.1) can be simplified as

$$\frac{dH_t}{dt} = -\sqrt{2H_t} \frac{1}{2\pi A} \iint_{\Pi} c(x)m(x)dx. \quad (5.2)$$

And (5.2) has an explicit solution

$$H_t = \left(\sqrt{H_0} - \frac{t}{2\sqrt{2\pi A}} \iint_{\Pi} c(x)m(x)dx \right)^2, \quad 0 \leq t \leq \frac{2\pi A(\sqrt{2H_0} - \sqrt{2H(O)})}{\iint_{\Pi} c(x)m(x)dx}. \quad (5.3)$$

Lemma 5.5 For any $\varepsilon > 0$, any $\delta > 0$ the invariant measure for the process $\Lambda_t^{\varepsilon, \delta}$ is proportional to the Lebesgue measure on Λ .

Lemma 5.6 For any $\delta > 0$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} P\{N_t^{\varepsilon, \delta} \text{ finally falls into well } 1\} &= \lim_{\varepsilon \downarrow 0} P\{\widehat{N}_t^{\varepsilon, \delta} \text{ finally falls into well } 1\}, \\ \lim_{\varepsilon \downarrow 0} P\{N_t^{\varepsilon, \delta} \text{ finally falls into well } 2\} &= \lim_{\varepsilon \downarrow 0} P\{\widehat{N}_t^{\varepsilon, \delta} \text{ finally falls into well } 2\}. \end{aligned}$$

At the beginning of this section we have already defined the process $Y_t^{\varepsilon,\delta} = (H_t^{\varepsilon,\delta}, K_t^{\varepsilon,\delta})$ on the tree Γ . Let Π_i , $i \in \{1, 2\}$ be the set of those points on Π which correspond to well i , $i \in \{1, 2\}$, respectively (see Fig.11, but notice that it only shows the s coordinate in the space Π).

Lemma 5.7

$$\begin{aligned} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{N_t^{\varepsilon,\delta} \text{ finally falls into well 1} \} &= \frac{\iint_{\Pi_1} c(s, \theta) m(s, \theta) ds d\theta}{\iint_{\Pi} c(s, \theta) m(s, \theta) ds d\theta}, \\ \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{N_t^{\varepsilon,\delta} \text{ finally falls into well 2} \} &= \frac{\iint_{\Pi_2} c(s, \theta) m(s, \theta) ds d\theta}{\iint_{\Pi} c(s, \theta) m(s, \theta) ds d\theta}. \end{aligned}$$

Now similarly as in Section 2 and Section 3, we consider a piecewise linear modification of the process $Y_t^{\varepsilon,\delta}$, and denote such a process by $\hat{Y}_t^{\varepsilon,\delta}$. We have $\hat{Y}_t^{\varepsilon,\delta} = (\hat{H}_t^{\varepsilon,\delta}, K_t^{\varepsilon,\delta})$. For all $\varepsilon > 0$, for all $0 < t < T$, for some uniform constant $C > 0$, $P(|\hat{Y}_t^{\varepsilon,\delta} - Y_t^{\varepsilon,\delta}| < C\varepsilon) = 1$. We define a stochastic process Y_t on Γ as follows: it has deterministic motion within each edge of the tree Γ and only has stochasticity at the interior vertex O . Let $Y_t = (H(t), K(t))$ where $H(t)$ is the energy of the particle and $K(t)$ is the number of the edge of the graph Γ . The deterministic motion within each edge of the tree Γ is defined as follows: on edge 3 the first component $H(t)$ of Y_t satisfies

$$H_t = \left(\sqrt{H_0} - \frac{t}{2\sqrt{2}\pi A} \iint_{\Pi} c(x) m(x) dx \right)^2, \quad 0 \leq t \leq t_0, \quad (5.4)$$

and on edge 1 and 2 the differential equation becomes

$$H_t = \left(\sqrt{H(O)} - \frac{t - t_0}{2\sqrt{2}\pi A_1} \iint_{\Pi_1} c(x) m(x) dx \right)^2, \quad t_0 \leq t \leq t_1, \quad (5.5)$$

$$H_t = \left(\sqrt{H(O)} - \frac{t - t_0}{2\sqrt{2}\pi A_2} \iint_{\Pi_2} c(x) m(x) dx \right)^2, \quad t_0 \leq t \leq t_2, \quad (5.6)$$

where Π_1 and Π_2 are defined as before, and A_1 , A_2 are the areas of the domains in G corresponding to well 1 and 2. The process Y_t , starting from a point $Y_0 = (H_0, 3)$, will reach the interior vertex O in time $t_0 = \frac{2\pi A(\sqrt{2H_0} - \sqrt{2H(O)})}{\iint_{\Pi} c(x) m(x) dx} < \infty$ and will instantaneously

leave O and enter one of the edges 1 or 2 with probabilities $p_1 = \frac{\iint_{\Pi_1} c(x) m(x) dx}{\iint_{\Pi} c(x) m(x) dx}$

and $p_2 = \frac{\iint_{\Pi_2} c(x)m(x)dx}{\iint_{\Pi} c(x)m(x)dx}$ respectively. After Y_t enters edge 1 or 2 it will move deterministically according to the given differential equation (5.5) or (5.6) till time $t_i = t_0 + \frac{2\pi A_i \sqrt{2H(O)}}{\iint_{\Pi_i} c(x)m(x)dx}$, $i \in \{1, 2\}$, respectively, when it hits energy level zero.

These lemmas imply the following:

Theorem 5.1 *The process $\hat{Y}_{t/\varepsilon}^{\varepsilon, \delta}$ converges weakly, as first $\varepsilon \downarrow 0$ then $\delta \downarrow 0$, to Y_t .*

Let us prove now these lemmas.

Proof of Lemma 5.1: Let τ be the first time for process I_t^θ to exit from the interval $[0, \pi]$. By considering $u(\theta) = \mathbb{E}_\theta \tau$ as the solution of the corresponding Dirichlet problem, one can check that $u(\theta)$ is finite so that the boundary points $\theta = 0, \pi$ are accessible.

Now we consider another diffusion process \tilde{I}_t^θ which starts from point θ and is governed by the operator $\tilde{\mathcal{L}}f = \frac{1}{2} \frac{d}{d\theta} (a(\theta) \frac{df}{d\theta})$ with instantaneous reflection at the boundary points $\theta = 0, \pi$. The invariant density for the process \tilde{I}_t^θ is the uniform distribution with the density $\frac{1}{\pi}$.

Our process I_t^θ can be obtained from \tilde{I}_t^θ by taking a time change $dt = \sin \theta d\tilde{t}$. Therefore invariant measure for I_t^θ is $\frac{1}{2} \sin \theta d\theta$ (see, [8], Chapter 5 for a reference concerning the random time change). ■

Proof of Lemma 5.2: For any small $r > 0$, any $\delta > 0$, any $\theta_1, \theta_2 \in [r, \pi - r]$, we have $p_I(\delta; \theta_1, \theta_2) \geq a(r) > 0$ for a positive constant $a(r) > 0$ depending on r , where $p_I(\cdot; \cdot, \cdot)$ is the transition density for the process I_t^θ .

Let $r > 0$. Let $\Pi(r) = \{x = (s, \theta) \in \Pi; r \leq \theta \leq \pi - r\}$. Since the function $\Theta(x) \equiv \Theta(s, \theta)$ is continuous for $x \in \Pi(r)$ and the set $\Pi(r)$ is compact, $\Theta(x)$ must attain its maximum and minimum on $\Pi(r)$. It is easy to see that there exist $\lambda(r) > 0$ such that $\min(r, \pi - \max_{x \in \Pi(r)} \Theta(x), \min_{x \in \Pi(r)} \Theta(x)) = \lambda(r) > 0$ (otherwise the particle must start from the tangential direction and keep on moving along the boundary, which contradicts the fact that $\theta \in [r, \pi - r]$). This implies that $f(\Pi(r)) \subseteq \Pi(\lambda(r))$ with $0 < \lambda(r) \leq r$.

We check the following form of Doeblin condition: For any $x \in \Pi(r)$, there exist an integer $n_0 \geq 1$ and a positive $\varepsilon > 0$ such that for any Borel measurable function A in $\mathcal{B}(\Pi(r))$, $\nu(A) \leq \varepsilon$, we have $p^{(n_0)}(x, A) \leq 1 - \varepsilon$. Here $\nu(\cdot)$ is the standard Lebesgue measure on Π .

Let $x = (s_1, \theta_1) \in \Pi(r)$. Let $(s_2, \theta_2) = (S(s_1, \theta_1), \Theta(s_1, \theta_1)) \in \Pi(\lambda(r))$. Let a

point $(s_3, \theta_3) \in \Pi(r)$, $s_3 \neq s_2$. Let $\zeta(s_2, s_3) \in (0, \pi)$ be the unique angle such that $S(s_2, \zeta(s_2, s_3)) = s_3$. Let the set $W = \{S(s_2, \theta); \theta \in [\lambda(r), \pi - \lambda(r)]\}$. We choose $\varepsilon_1 > 0$ such that $\nu(\Pi(r)) - \varepsilon_1 \geq \varepsilon_2 > 0$. Then for $\nu(A) \leq \varepsilon_1$ we have $\nu(\Pi(r) \setminus A) \geq \varepsilon_2$. We first consider the case when $\nu((W \times [r, \theta - r]) \cap (\Pi(r) \setminus A)) \geq \varepsilon_3 > 0$. This implies that there exist $\varepsilon_4 > 0$ such that $\text{mes}(\{s_3 \in W; (s_3, \theta_3) \in \Pi(r) \setminus A \text{ for some } \theta_3 \in [r, \pi - r]\}) \geq \varepsilon_4 > 0$. Here $\text{mes}(\cdot)$ is the standard Lebesgue measure on \mathbb{R} . We have, for some $\varepsilon_5 = \varepsilon_5(\lambda(r)) > 0$ depending on $\lambda(r)$, that $\left| \frac{\partial \zeta}{\partial s_3} \right| \geq \varepsilon_5 > 0$ for $s_3 \in W$. Let $K = \{\zeta(s_2, s_3); s_3 \in W, (s_3, \theta_3) \in \Pi(r) \setminus A \text{ for some } \theta_3 \in [r, \pi - r]\}$. We see that $\text{mes}(K) \geq \varepsilon_4 \varepsilon_5 > 0$. We also notice that $K \subseteq [\lambda(r), \pi - \lambda(r)]$. Let us define the set $M = \{\theta_3 \in [r, \pi - r]; (s_3, \theta_3) \in \Pi(r) \setminus A \text{ for some } s_3 \in W\}$. There exist $\varepsilon_6 > 0$ such that $\text{mes}(M) \geq \varepsilon_6 > 0$. Now we have

$$\begin{aligned} p^{(2)}(x, \Pi(r) \setminus A) &= \int_{\theta_3 \in M} \int_{\theta \in K} p_I(\delta; \theta_2, \theta) p_I(\delta; \Theta(s_2, \theta), \theta_3) d\theta d\theta_3 \\ &\geq \int_{\theta_3 \in M} \int_{\theta \in K} a(\lambda(r)) a(\lambda(\lambda(r))) d\theta d\theta_3 \\ &\geq a(\lambda(r)) a(\lambda(\lambda(r))) \varepsilon_4 \varepsilon_5 \varepsilon_6 > 0. \end{aligned}$$

Therefore $p^{(2)}(x, A) \leq 1 - a(\lambda(r)) a(\lambda(\lambda(r))) \varepsilon_4 \varepsilon_5 \varepsilon_6$. So in this case we can choose $n_0 = 2$ and $0 < \varepsilon < \min(a(\lambda(r)) a(\lambda(\lambda(r))) \varepsilon_4 \varepsilon_5 \varepsilon_6, \varepsilon_1)$.

Now let us consider the case when $\nu((W \times [r, \theta - r]) \cap (\Pi(r) \setminus A)) = 0$. Since we have $\nu(\Pi(r) \setminus A) \geq \varepsilon_2 > 0$, we have $\nu((([0, L] \setminus W) \times [r, \theta - r]) \cap (\Pi(r) \setminus A)) \geq \varepsilon_2 > 0$. In this case we can take one more transition from (s_2, θ_2) to some other set of positive measure lying away from $[0, L] \setminus W$ (provided that r is small enough) and carry out a similar argument as above. We conclude in this case that $n_0 = 3$.

It is easy to see that this chain has only one ergodic component $\Pi(r)$, since from any point $x \in \Pi(r)$ the chain can come to any neighborhood of any other point $y \in \Pi(r)$ in several steps.

Since $m(x)$ is an invariant density for both the billiard map f and the diffusion process I_t^θ , it is also invariant for the chain $\{X_n^\delta\}_{n \geq 1}$. Therefore, after normalization, the unique invariant density for the chain $\{X_n^\delta\}_{n \geq 1}$ is $d(x) = \frac{1}{2L} m(x)$. ■

In particular, Law of Large Numbers holds for this Markov chain $\{X_n^\delta\}_{n \geq 1}$ on Π .

Proof of Lemma 5.3: Suppose there are $n = n(\varepsilon)$ collisions happened during time $[\frac{t}{\varepsilon}, \frac{t + \Delta t}{\varepsilon})$, and the corresponding successive positions of the particle on Π are $X_{k+1}^\delta, \dots, X_{k+n}^\delta$. It is clear that $n \rightarrow \infty$ as $\varepsilon \downarrow 0$ and $n\varepsilon \sim O(\Delta t)$ as $\varepsilon \downarrow 0$. For small $\Delta t > 0$, we have

$$H_{(t+\Delta t)/\varepsilon}^{\varepsilon,\delta} - H_{t/\varepsilon}^{\varepsilon,\delta} = -\Delta t \left(\frac{\Delta t}{n\varepsilon}\right)^{-1} \frac{c(X_{k+1}^\delta) + c(X_{k+2}^\delta) + \dots + c(X_{k+n}^\delta)}{n}.$$

We also have

$$\frac{\Delta t}{n\varepsilon} = \frac{t_{\text{initial}} - t_{\text{final}} + T(X_{k+1}^\delta, H_1) + T(X_{k+2}^\delta, H_2) + \dots + T(X_{k+n}^\delta, H_n)}{n}, \quad (5.7)$$

where H_i , $1 \leq i \leq n$ are the energies of the particle at the i -th collision (before it loses some energy); t_{initial} is the time between the starting time $\frac{t}{\varepsilon}$ of the motion and the time of the first collision with ∂G ; t_{final} is the time between the ending time $\frac{t + \Delta t}{\varepsilon}$ of the motion and the time after that when the particle has next (the $n + 1$ -th) collision with the boundary of G . Since function $T(x, H)$ is Lipschitz in H (when $H > h > 0$) with respect to x , we have $|T(x, H_i) - T(x, H_{t/\varepsilon}^\varepsilon)| \leq C|H_i - H_{t/\varepsilon}^\varepsilon| \sim O(\Delta t)$ as $\varepsilon \downarrow 0$. Therefore from (5.2), by using the Law of Large Numbers for this Markov chain, we get

$$\frac{\Delta t}{n\varepsilon} \rightarrow \mathbb{E}T(X, H_{t/\varepsilon}^\varepsilon) + O(\Delta t)$$

in probability, as $\varepsilon \downarrow 0$.

By using the Law of Large Number for our Markov chain we also see that

$$\frac{c(X_{k+1}^\delta) + c(X_{k+2}^\delta) + \dots + c(X_{k+n}^\delta)}{n} \rightarrow \mathbb{E}c(X)$$

in probability, as $n \rightarrow \infty$.

Here the random variable X has the stationary distribution of this chain on Π with density $d(x) = \frac{1}{2L}m(x)$. Therefore

$$\lim_{\varepsilon \downarrow 0} \frac{H_{(t+\Delta t)/\varepsilon}^{\varepsilon,\delta} - H_{t/\varepsilon}^{\varepsilon,\delta}}{\Delta t} = -\frac{\mathbb{E}c(X)}{\mathbb{E}T(X, H_{t/\varepsilon}^\varepsilon) + O(\Delta t)}$$

in probability.

This means that as $\varepsilon \downarrow 0$, the rescaled process $H_{t/\varepsilon}^{\varepsilon,\delta}$ within edge I_3 of the tree Γ converges in probability to the trajectory of the deterministic process defined by the differential equation

$$\frac{dH_t^\delta}{dt} = -\frac{\int \int_{\Pi} c(x)m(x)dx}{\int \int_{\Pi} T(x, H_t)m(x)dx}.$$

Notice that this is independent of δ . Therefore letting $\delta \downarrow 0$, we get the desired result. ■

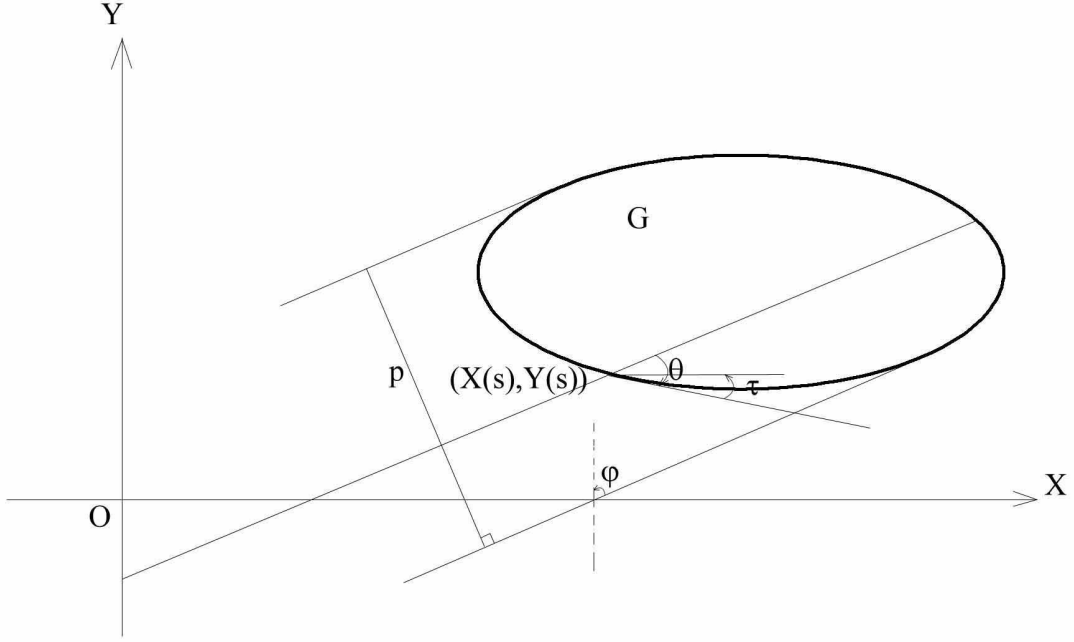


Fig. 12: A problem in integral geometry

Proof of Lemma 5.4: This is a known result in integral geometry (see [11]). But we will still provide a short proof here in order for the reader to understand some calculations later in this section.

Let the equation of the line intersecting the closed curve ∂G be written in the form (see Fig.12)

$$X \cos \varphi + Y \sin \varphi = p .$$

Let the curve ∂G be

$$X = X(s), Y = Y(s)$$

where s is the arc length parameter.

Let τ be the angle of the tangential direction at point (X, Y) , we have

$$\cos \tau = X'(s) , \sin \tau = Y'(s) , \frac{\pi}{2} + \varphi - \tau = \theta .$$

We then have

$$d\varphi = d\theta + d\tau = d\theta + \kappa ds .$$

Here $\kappa = \kappa(X, Y)$ is the curvature at point (X, Y) , and

$$\begin{aligned}
dp &= X'(s) \cos \varphi ds - X(s) \sin \varphi d\varphi + Y'(s) \sin \varphi ds + Y(s) \cos \varphi d\varphi \\
&= (X'(s) \cos \varphi + Y'(s) \sin \varphi) ds - X(s) \sin \varphi (\kappa ds + d\theta) + Y(s) \cos \varphi (\kappa ds + d\theta) \\
&= (X'(s) \cos \varphi + Y'(s) \sin \varphi + \kappa(-X(s) \sin \varphi + Y(s) \cos \varphi)) ds + (-X(s) \sin \varphi + Y(s) \cos \varphi) d\theta .
\end{aligned}$$

These equalities imply that

$$\begin{aligned}
dp \wedge d\varphi &= (X'(s) \cos \varphi + Y'(s) \sin \varphi + \kappa(-X(s) \sin \varphi + Y(s) \cos \varphi)) ds \wedge d\theta \\
&\quad + \kappa(-X(s) \sin \varphi + Y(s) \cos \varphi) d\theta \wedge ds \\
&= (X'(s) \cos \varphi + Y'(s) \sin \varphi) ds \wedge d\theta \\
&= (\cos \tau \cos \varphi + \sin \tau \sin \varphi) ds \wedge d\theta \\
&= \cos(\tau - \varphi) ds \wedge d\theta = \sin \theta ds \wedge d\theta .
\end{aligned}$$

So we have that

$$\int_0^\pi \int_0^L L(s, \theta) \sin \theta ds \wedge d\theta = 2 \int_0^\pi \int_0^P L(s, \theta) dp \wedge d\varphi = 2\pi A.$$

We have used the fact that $\int_0^P L(s, \theta) dp = A$, where P is the height of G projected onto the direction perpendicular to $l(s, \theta)$ (recall that $l(s, \theta)$ is the chord starting from $(s, \theta) \in \Pi$). ■

We would like to mention that the relation $dp \wedge d\varphi = \sin \theta ds \wedge d\theta$ will be used in some of our later calculations. In future calculations, we will use the abbreviated form $dpd\varphi = \sin \theta dsd\theta$.

Proof of Lemma 5.5: From Lemma 5.2 we already know that the embedded chain $\{X_n^\delta\}_{n \geq 1}$ has a unique invariant measure on Π . Since our process $\Lambda_t^{\varepsilon, \delta}$ is purely deterministic within $\Lambda \setminus \Pi$, and any point in $\Lambda \setminus \Pi$ can be reached in finite time from a point on Π , $\Lambda_t^{\varepsilon, \delta}$ must be positive recurrent on Λ and therefore it has a unique invariant measure on Λ .

We use the Khasminskii formula (see, [6], on page 185, formula (4.1)) for the expression (up to a constant factor) of the invariant measure $\hat{\mu}(\cdot)$ for $\Lambda_t^{\varepsilon, \delta}$ on Λ through the invariant measure $\frac{1}{2L}m(x)dx$ for the embedded chain $\{X_n^\delta\}_{n \geq 1}$ on Π : for any measurable set $B \subseteq \Lambda$,

$$\hat{\mu}(B) = \frac{1}{2L} \iint_{\Pi} m(x) dx \mathbb{E}_x \int_0^{\tau_1^x} \chi_B(\Lambda_t^{\varepsilon, \delta}) dt . \quad (5.8)$$

Here τ_1^x is the first time when $\Lambda_t^{\varepsilon, \delta}$, $\Lambda_0^{\varepsilon, \delta} = x \in \Pi$, reaches Π again. Note that for a given x , τ_1^x as well as the motion $\Lambda_t^{\varepsilon, \delta}$, $0 < t < \tau_1^x$, and in particular, the time spent by

$\Lambda_t^{\varepsilon, \delta}$ in B , are not random, so that the expectation sign in (5.8) can be omitted. Since the invariant measure of our imbedded chain and of the billiard map are the same, we conclude from (5.8) that $\hat{\mu}(B)$ coincides with the invariant measure of the non-perturbed billiard. As is known, the Lebesgue measure is invariant for the billiard. Thus $\hat{\mu}(B)$ is the uniform distribution on Λ . ■

Proof of Lemma 5.6: This follows from the fact that

$$P\{\max_{0 \leq t < T < \infty} |H_t^{\varepsilon, \delta} - \hat{H}_t^{\varepsilon, \delta}| < C\varepsilon\} = 1$$

where $C > 0$ is a constant. ■

Proof of Lemma 5.7: Consider a time change in the process $\hat{N}_t^{\varepsilon, \delta}$. If at time t , $\Lambda_t^{\varepsilon, \delta}$ lies on a chord $l(x)$ starting from $x = (s, \theta) \in \Pi$, then we set $d\tilde{t} = \frac{L(x)}{c(f(x))}dt$. Here $f(x) = f(s, \theta) = (S(s, \theta), \Theta(s, \theta))$ is the billiard map and $L(x)$ is the length of the chord $l(x)$. The density of the invariant measure for $\Lambda_t^{\varepsilon, \delta}$ (with respect to Lebesgue measure) is proportional to $\frac{c(f(x))}{L(x)}$. Here $x = x(g, \varphi)$ is a point in Π such that $(g, \varphi) \in \Lambda$ lies on the chord $l(x)$ starting from x . This time change corresponds to multiplication of the generator of the process $\hat{N}_t^{\varepsilon, \delta}$ by $\frac{L(x)}{c(f(x))}$.

The H -component of the process obtained after time change moves down uniformly with a non-random speed, so that it hits level $H(O)$ at a non-random time $t = t(H_0)$. Let $\Lambda_1 = \{(g, \varphi) \in \Lambda : f(x(g, \varphi)) \in \Pi_1\}$, $\Lambda_2 = \{(g, \varphi) \in \Lambda : f(x(g, \varphi)) \in \Pi_2\}$. Here Π_1, Π_2 are defined as before, i.e., Π_i , $i \in \{1, 2\}$ is the set of those points on Π which correspond to well i , $i \in \{1, 2\}$.

By using Lemma 5.5, we see that

$$\lim_{\varepsilon \downarrow 0} P\{\hat{N}_t^{\varepsilon, \delta} \text{ finally falls into well 1}\}$$

$$\begin{aligned} &= D \iiint_{\Lambda_1} \frac{c(f(x(g, \varphi)))}{L(x(g, \varphi))} dg d\varphi \\ &= D \iiint_{\Lambda_1} \frac{c(f(x(g, \varphi)))}{L(x(g, \varphi))} dL(x(g, \varphi)) dp d\varphi \\ &= D \iint_{f^{-1}(\Pi_1)} \frac{c(f(s, \theta))}{L(s, \theta)} \left(\int_0^{L(s, \theta)} dL(s, \theta) \right) \sin \theta ds d\theta \\ &= D \iint_{f^{-1}(\Pi_1)} c(f(s, \theta)) m(s, \theta) ds d\theta \\ &= D \iint_{\Pi_1} c(s, \theta) m(s, \theta) ds d\theta. \end{aligned}$$

Here $m(s, \theta) = \sin \theta$ is the density of the Liouville measure. We have used the fact

that $dg d\varphi = dL(s, \theta) dp d\varphi = dL(s, \theta) \sin \theta ds d\theta$ (notice here that $dg = dL dp$), as was calculated in Lemma 5.4, and the invariance of Liouville measure under the billiard map. Normalizing constant $D = \left(\iiint_{\Lambda} \frac{c(f(x(g, \varphi)))}{L(x(g, \varphi))} dg d\varphi \right)^{-1} = \left(\iint_{\Pi} c(s, \theta) m(s, \theta) ds d\theta \right)^{-1}$.

Similarly, we have $\lim_{\varepsilon \downarrow 0} P\{\widehat{N}_t^{\varepsilon, \delta} \text{ finally falls into well 2}\} = \frac{\iint_{\Pi_2} c(s, \theta) m(s, \theta) ds d\theta}{\iint_{\Pi} c(s, \theta) m(s, \theta) ds d\theta}$.

The trajectory of $\widehat{N}_t^{\varepsilon, \delta}$ is the same as $\widehat{N}_t^{\varepsilon, \delta}$. Taking into account this fact and Lemma 5.6, the statement follows. ■

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